Theorem: $X^{\text{top space}}$. $A^{\text{subset}} \subseteq X$. $\mathcal{U}$ = collection of open sets U of X s/t $A \cap U$ is meager. Then $A \cap \overline{(\bigcup_{U \in \mathcal{U}} U)}$ is meager

Observations before proof: ① U^{open}, A^{subset} & $A \cap U$ nowhere dense $\Rightarrow$ $A \cap \overline{U}$ nowhere dense

Proof: $A \cap \overline{U} \subseteq (\overline{U} \setminus U) \cup (A \cap U)$ both of which are nowhere dense

② U^{open}, A^{subset} & $A \cap U$ meager $\Rightarrow$ $A \cap \overline{U}$ meager

Proof: (same as above) $A \cap \overline{U} \subseteq \underbrace{(A \cap (\overline{U} \setminus U))}_{\text{nowhere dense}} \cup \underbrace{(A \cap U)}_{\text{meager}}$.

③ Suppose $\{U_\alpha\}_{\alpha \in I}$ be a family of pairwise disjoint open sets. Let $\{N_\alpha\}_{\alpha \in I}$ be a family of nowhere dense sets, $N_\alpha \subseteq U_\alpha$. Then $\bigcup N_\alpha$ is nowhere dense. [Proof: Suppose $\phi \neq V^{\text{open}} \subseteq \overline{\bigcup N_\alpha}$. Then $V \cap N_\alpha \neq \phi$ for some α_0, so $\phi \neq V \cap U_{\alpha_0}^{\text{open}}$ is open ~~for that α_0~~

Now $\phi \neq V \cap U_{\alpha_0} \subseteq \overline{\bigcup N_\alpha} \cap U_{\alpha_0} \subseteq \overline{(\bigcup N_\alpha) \cap U_{\alpha_0}} = \overline{N_{\alpha_0}}$.]

$\because U_{\alpha_0}$ is open

COR: A non-meager $\Rightarrow$ $\exists\, V^{\text{open}}_{\phi \neq}$ s/t $\forall x \in \overline{V}$ $\forall$ U^{nbhd} of x, we have $U \cap A$ is non-meager.

Proof: Suppose not. Then $\forall_{\phi \neq} V^{\text{open}}$ $\exists$ $x_V \in \overline{V}$ & U_V^{nbhd} of x_V s/t $U_V \cap A$ is non-meager.

By theorem $\overline{(\bigcup_V U_V)} \cap A$ is non-meager. ETS $\overline{\bigcup_V U_V} = X$

Proof of this claim: $y \notin \overline{\bigcup_V U_W}$. Let W be open around y s/t $W \cap \bigcup_V U_V = \phi$. But observe $W \cap U_W$ is non-empty by construction. QED

Proof (of the theorem) We first show: Suppose $\mathcal{U}'$ is a collection of ~~disjo~~ pairwise disjoint open sets s/t $A \cap U$ is meager for all U in $\mathcal{U}'$. We show $A \cap \overline{(\bigcup_{U \in \mathcal{U}'} U)}$ is meager

Proof of this statement: By observation ②, ETS $A \cap (\bigcup_{U \in \mathcal{U}'} U)$ is meager.

Write $A \cap U = \bigcup_{i=1}^{\infty} N_i^U$ with N_i^U nowhere dense. Now $A \cap (\bigcup_{U \in \mathcal{U}'} U) = \bigcup_{U \in \mathcal{U}'} (A \cap U)$

$= \bigcup_{U \in \mathcal{U}'} \bigcup_{i=1}^{\infty} N_i^U = \bigcup_{i=1}^{\infty} \underbrace{(\bigcup_{U \in \mathcal{U}'} N_i^U)}$ nowhere dense by observation ③.