FOR METRIC SPACES ONLY (BAIRE CATEGORY THEOREM)

Generalization to complete uniform spaces is not known (unfortunately)

Thm (BAIRE) X locally compact regular/Complete pseudo metric space. Then the ~~countable~~ intersection of any countable family of dense open sets is itself dense.

Proof: U arbitrary non-empty open set, $\{G_n ; n \geq 0\}$ family of dense open sets.

Put $V_0 = U$. Choose inductively non-empty open sets V_i as follows:

$$\left.\begin{array}{ll} V_1 \text{ is relatively compact s/t } & \overline{V_1} \subseteq V_0 \cap G_0 \\ V_2 \text{ — " — } & \text{s/t } \overline{V_2} \subseteq V_1 \cap G_1 \\ \vdots & \end{array}\right\} \text{locally compact regular case}$$

$$\left.\begin{array}{l} \overline{V_1} \subseteq V_0 \cap G_0 \text{ s/t } \dim(V_1) \leq 1 \\ \overline{V_2} \subseteq V_1 \cap G_1 \text{ s/t } \dim(V_2) \leq \frac{1}{2} \\ \vdots \end{array}\right\} \text{complete pseudo-metric case}$$

Claim: $\cap V_i$ is non-empty. Since $V_0 \subseteq U$, ~~this proves~~ and $\cap V_i \subseteq \cap G_i$ by construction, this proves $U \cap \cap G_i$ is non-empty.

Proof of claim: $\cap V_i = \cap \overline{V_i}$ by construction. In the LC+ regular case, $\overline{V_i}$ is a family of closed subsets of the compact set $\overline{V_1}$ & this family has FIP; in the complete p.m. case, choose $v_i \in V_i$; then $\{v_i\}$ is Cauchy, say v is a limit. Now $v \in \cap \overline{V_i}$. Since $v_j \in V_i \ \forall \ j \geq i$. QED.

Ex: Non-empty locally compact Hdf space without isolated points is uncountable.

RESTATEMENT: Complement of a Meager set in a complete metric space is ~~[illegible]~~ dense.

Nowhere dense: Closure has empty interior

Meager: Countable union of nowhere dense sets.

" First category.

Observations: • If U is open then $\overline{U} \setminus U$ is nowhere dense

• Closure of a nowhere dense set is nowhere dense.

• Nowhere dense iff complement of the closure is open dense

• Finite ~~intersection~~ unions of nowhere dense sets are nowhere dense:

$$X \setminus \left(\overline{N_1 \cup N_2}\right) = X \setminus \left(\overline{N_1} \cup \overline{N_2}\right) = \left(X \setminus \overline{N_1}\right) \cap \left(X \setminus \overline{N_2}\right)$$

Finite intersections of open dense sets is open dense.

• A^{subset} & U^{open}. Then $A \cap U$ is nowhere dense $\Rightarrow \overline{A} \cap U$ is nowhere dense $[\because \overline{A} \cap U \subseteq \overline{A \cap U}]$

• $A^{\text{subset}}, U_\alpha^{\text{open}}$ & $A \cap U_\alpha$ is nowhere dense $\forall \alpha \Rightarrow A \cap \cup U_\alpha$ is nowhere dense

Proof: Suppose not. Choose $V^{\text{open}}_{\neq \phi} \subseteq \overline{A \cap (\cup U_\alpha)} \Rightarrow V \cap U_\alpha \neq \phi$ for some α

So: $\phi \neq V \cap U_\alpha \subseteq \overline{A \cap (\cup U_\alpha)} \cap U_\alpha \subseteq \overline{A \cap U_\alpha}$ $[\because \overline{B} \cap W \subseteq \overline{B \cap W}$ for $W^{\text{open}}]$.

QED