Proof (Continued) $\Leftarrow$ Suppose we have a Cauchy net $\{S_n \mid n \in D\}$. Consider the family $\{S_n \mid n \geq N\}$ as N varies over D. This collection has FIP. It contains small sets since the net is Cauchy. Now, by hypothesis, $\bigcap_N \overline{\{S_n \mid n \geq N\}}$ is non-empty. A point belonging to the intersection is a cluster point of the Cauchy net. So the net converges to it. QED

The following is NOT true: if, in a first countable uniform space, every Cauchy sequence converges, then ~~the space is so~~ is the space complete? However as a corollary of the characterization above we can prove:

COROLLARY: In a pseudo-metrizable uniform space, if every Cauchy sequence converges, then the space is complete.

Proof: $\Leftarrow$ trivial. $\Rightarrow$ We'll show that the characterization holds. Let Ω be a family of closed sets with FIP & containing small sets. Let $A_i \in \Omega$ s/t $\operatorname{diam}(A_i) \leq \frac{1}{n}$. Let $x_i \in A_i$. Then $\{x_i\}$ is Cauchy.
Proof: given $\varepsilon > 0$ choose N s/t $2/N < \varepsilon$. Let $n, m \geq N$. Choose $y \in A_n \cap A_m$.
Then $d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon$. ∎
Then x be the limit of this sequence. For every $A \in \Omega$, note $d(A, x_n) < \frac{1}{n}$ (because $A_n \cap A$ is non-empty). So x belongs to A ($\because A$ is closed). ∎

Standard method of proving completeness: show that the space is uni. isomorphic to a closed subspace of a product of complete spaces ~~and invoking the~~ which by the following result is complete:

Theorem: ~~The~~ product of complete uniform spaces is complete. Conversely, if a product is complete of a family of uniform spaces none of which is empty, then each of the spaces in the family is complete.

Propn: A net in a product is Cauchy iff its projection to each factor is so.
Proof: $\because$ it is the initial uniformity. The pull-backs of gages of each factor generate the gage of the product. ∎