Completeness (of uniform spaces).

Terminology: Gage: The Collection of pseudo-metrics that are u. cont w.r.t. $\mathcal{U}$.

Defn: A net $\{S_n \mid n \in D\}$ in X is Cauchy if $\forall\, U \in \mathcal{U}\ \exists\, N \in D$ s/t $\forall\, m,n \geq N$ ~~then~~ $(S_m, S_n) \in U$. Other equivalent formulations:

- $\forall$ P in the gage & $\varepsilon > 0$, $\{(S_m, S_n) \mid (m,n) \in D \times D\}$ is eventually in $V_{P,\varepsilon}$ where $V_{P,\varepsilon} := \{(x,y) \mid P(x,y) < \varepsilon\}$.
- $\forall\, U \in \mathcal{U}$, $\{(S_m, S_n) \mid (m,n) \in D \times D\}$ is eventually in U.
- $\forall$ U in a subbase of $\mathcal{U}$, $\{(S_m, S_n) \mid (m,n) \in D \times D\}$ is eventually in U.
- $\forall$ P in a generating collection of pseudo-metrics, $\{P(S_m, S_n) \mid (m,n) \in D \times D\}$ converges to 0.

Propn: A converging net is Cauchy. A Cauchy net converges to each of its cluster points.

Def: A uniform space is complete if every Cauchy net converges.

Propn: A closed subspace of a complete space is complete. A complete subspace of a Hdf uniform space is closed.

Examples of complete uniform spaces: ① $\mathcal{U}$ = all supsets of Δ ② $\mathcal{U} = \{\Delta, X \times X\}$ ③ Compact uniform spaces ④ $\mathbb{R}$ with its usual uniformity.

A characterization of completeness (reminiscent of a characterisation of compactness)

Exercise: The following are equivalent for a collection $\mathcal{O}$ of subsets of ~~X~~ of a uniform space X: (if these are satisfied $\mathcal{O}$ is said to contain small sets). ① $\forall\, U \in \mathcal{U}\ \exists\, A \in \mathcal{O}$ s/t $A \subseteq U[x]$ for some $x \in X$ ② $\forall\, U \in \mathcal{U}$, $\exists\, A \in \mathcal{O}$ s/t $A \times A \subseteq U$ ③ $\forall$ P p.metric u. cont w.r.t $\mathcal{U}$ & $\forall\, \varepsilon > 0$, $\exists\, A \in \mathcal{O}$ s/t the P-diameter of A is less than ε.

Theorem: A uniform space is complete iff for ~~eve~~ the intersection is non-empty of any family of closed sets with the finite intersection property & having small sets.

Proof: $\Rightarrow$. Finite subsets of the index set of the family form a directed set, say D. ~~We~~ By taking an elt belonging to the intersection of the subsets we get a net with index set D. The net is Cauchy $\because$ the family has small sets. The limit of the net belongs to each closed set in the family.