Proof of the metrization lemma: Define a real valued function $f: X \times X \to \mathbb{R}_{\geq 0}$ by $f(x,y) = 1/2^n$ if $(x,y) \in U_{n-1} \setminus U_n$ and $f(x,y) = 0$ for $(x,y) \in \bigcap U_n$. The desired function d is constructed from its "first approximation" f by a chaining argument. For $(x,y) \in X \times X$, let $d(x,y)$ be the infimum of $\sum_{i=0}^{n} f(x_i, x_{i+1})$, ~~st~~ over all finite sequences ~~x~~ $= x_0, x_1, \dots, x_{n+1} = y$. Then

- d satisfies the triangle inequality & $-$ $f(x,y) \geq d(x,y)$, so $U_n \subseteq \{(x,y) \mid d(x,y) < \frac{1}{2^n}\}$.

If each U_n is symmetric, then d is symmetric and so a pseudo metric.

Claim: $f(x_0, x_{n+1}) \leq 2 \sum_{i=0}^{n} f(x_i, x_{i+1})$

Proof continued modulo claim: It follows from the claim that $d(x,y) \geq \frac{1}{2} f(x,y)$. So $d(x,y) < \frac{1}{2^n} \Rightarrow f(x,y) < \frac{1}{2^{n-1}}$ $\Rightarrow (x,y) \in U_{n-1}$ and the lemma is proven.

Proof of claim: By induction on n. For $n=0$, the claim is trivial. Suppose that $n \geq 1$. Set $a = \sum_{i=0}^{n} f(x_i, x_{i+1})$. If $a=0$, then each $(x_i, x_{i+1}) \in \bigcap U_n$ (and since $U_n \circ U_n \circ U_n \subseteq U_{n-1}$) this means $(x_0, x_{n+1}) \in \bigcap U_n$, so $rhs = 0$. Now suppose $a > 0$. Let k be ~~least largest s/t f~~ largest, $0 \leq k \leq n$, s/t $f(x_0, x_1) + \dots + f(x_{k-1}, x_k)$ is $\leq a/2$. Note that $f(x_{k+1}, x_{k+2}) + \dots + f(x_n, x_{n+1}) \leq \frac{a}{2}$. By induction, we get $f(x_0, x_k) \leq 2 \frac{a}{2} = a$; also $f(x_{k+1}, x_{n+1}) \leq a$. And $f(x_k, x_{k+1}) \leq a$ trivially.

Let m be least s/t $2^{-m} \leq a$. Then $f(x_0, x_k) \leq 2^{-m}$ (because f always takes a value 2^{-k}), $f(x_{k+1}, x_{n+1}) \leq 2^{-m}$ and $f(x_k, x_{k+1}) \leq 2^{-m}$. This means (x_0, x_k), (x_{k+1}, x_{n+1}), (x_k, x_{k+1}) all belong to U_{m-1}. Since $U_{m-1} \circ U_{m-1} \circ U_{m-1} \subseteq U_{m-2}$, we get $(x_0, x_{n+1}) \in U_{m-2}$, so $f(x_0, x_{n+1}) \leq \frac{1}{2^{m-1}} = 2 \cdot \frac{1}{2^m} \leq 2a$. QED