Uniform Continuity; Product Uniformities : $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous if $(f\times f)^{-1}(V) \in \mathcal{U}$ $\forall$ $V \in \mathcal{V}$. Equivalently, $(f\times f)^{-1}(V) \in \mathcal{U}$ for every V in a subbase of $\mathcal{V}$. Observe: if $X = Y = \mathbb{R}$ & $\mathcal{U} = \mathcal{V}$ = usual uniformity then Unif. cont. fns in the above sense are precisely those that are u. cont. in the sense that we learn in real analysis.

Uniform isomorphism; Uniform equivalent spaces.

Theorem: U. cont $\Rightarrow$ Cont. Proof: $f^{-1}(V[f x]) = (f\times f)^{-1}(V)[x]$. QED

Exercise: $X \xrightarrow{f} Y$ (set, uniform). $\{f^{-1}V \mid V \text{ entourage}\}$ is a base for a uniformity on X. This is the coarsest uniformity on X w.r.t. which f is u. cont.

Uniform subspace: Apply the exercise above to the inclusion of a subset. The uniform topology of a uniform subspace is the subspace top of the uniform topology.

Initial Uniformity: $X \xrightarrow{f_\alpha} X_\alpha$ (set, uniform spaces). $f_\alpha^{-1} U_\alpha$ as U_α runs over entourages of X_α and α over the index set is a subbase for a uniformity on X.

Particular case: uniform subspace; product uniformity

Theorem: The top of an initial uniformity is the initial top of the corr. uni. topologies (Exercise). In particular, the top of the product/subspace uniformity is the product/subspace of the uniform topologies.

Theorem: $(X,\mathcal{U})$ uniform space, and d is pseudo-metric on X. Then d is u. cont on $X\times X$ (for the product uniformity) iff $\{(x,y) \mid d(x,y) < \varepsilon\} \in \mathcal{U}$ $\forall$ $\varepsilon > 0$.

Proof: Set $V_{d,\varepsilon} := \{(x,y) \mid d(x,y) < \varepsilon\}$. $\widetilde{U}:=$

A base for the product uniformity on $X\times X$: $\{((x,y),(u,v)) \mid (x,u) \in U \ \& \ (y,v) \in U\}$ U in $\mathcal{U}$

Suppose d is u. cont. Fix $\varepsilon > 0$. $\exists$ U in $\mathcal{U}$ s/t $|d(x,y) - d(u,v)| < \varepsilon$ if $(x,u) \in U$ & $(y,v) \in U$. In particular, putting $u = v = y$, we get $d(x,y) < \varepsilon$ for $(x,y) \in U$. Thus $V_{d,\varepsilon} \supseteq U$, so $V_{d,\varepsilon} \in \mathcal{U}$.

Converse: Observe $\widetilde{V}_{d,\varepsilon/2} \subseteq (d)^{-1}(V_{d,\varepsilon})$ QED