Theorem: For U an entourage, the interior $\mathring{U}$ of U is also an entourage. Thus the open symmetric ~~members of U~~ entourages form a base for the uniformity.

Proof: Uses Lemma above. The argument is a generalization of "$\varepsilon/3$ arguments." Choose V symmetric entourage s/t $V \circ V \circ V \subseteq U$.
By lemma $V \circ V \circ V = \bigcup_{(x,y) \in V} V[x] \times V[y]$, so $V \subseteq \mathring{U}$, so $\mathring{U}$ is an entourage E

CoR: Every entourage is a nbhd of the diagonal.

Remark: Not every nbhd of the diagonal is an entourage (e.g., as already seen, $\{(x,y) \mid |x-y| < 1/1+x^2\}$ in $\mathbb{R}^2$). In fact, there could be many different uniformities (on a set X) all inducing the same topology and so having the same nbhd system of the diagonal.

Theorem: $A \overset{\text{subset}}{\subseteq} X$. Then $\overline{A} = \bigcap \{U[A] \mid U \text{ entourage}\}$
$M \overset{\text{subset}}{\subseteq} X \times X$. Then $\overline{M} = \bigcap \{U \circ M \circ U \mid U \text{ entourage}\}$.

Proof: ① $x \in \overline{A} \iff A$ meets every fund nbhd of $x \iff \forall\, U^{\text{symm}}\ \exists\, a \in A$ s/t $x \in U[a]$
$\iff x \in \bigcap_{U^{\text{symm}}} \underbrace{\bigcup_{a \in A} U[a]}_{= U[A]} \iff x \in \bigcap_{U^{\text{symm}}} U[A] \iff x \in \bigcap_{U \in \mathcal{U}} U[A]$.
$\because$ given U $\exists\, V^{\text{symm}}$ s/t $V \subseteq U$

② $(x,y) \in \overline{M} \iff M$ meets $U[x] \times U[y]\ \forall\, U^{\text{symm}} \iff \bigcap_{U^{\text{symm}}} \underbrace{\bigcup_{(m,n) \in M} U[m] \times U[n]}_{= U \circ M \circ U} \ni (x,y)$ QED

Theorem: The closed symm. members of $\mathcal{U}$ form a base for $\mathcal{U}$.

Proof: Given $U \in \mathcal{U}$, let $V \in \mathcal{U}$ s/t $V \circ V \subseteq U$. Choose $W' \in \mathcal{U}$ s/t $W' \circ W' \subseteq V$
Let $W = W' \cap W'^{-1}$. Then W is symmetric, in $\mathcal{U}$, & $W \circ W \subseteq V$. So $W \circ W \circ W \subseteq W \circ W \circ W \circ W$
$\subseteq V \circ V \subseteq U$ on the one hand; & $\overline{W} \subseteq W \circ W \circ W$ on the other. So $W \subseteq \overline{W} \subseteq U$. QED

Exercise: TFAE for a uniform space $(X, \mathcal{U})$. ① X Hausdorff ② X is T_1 (accessible)
③ $\bigcap_{U \in \mathcal{U}} U = \Delta$. ④ X regular. (First observe that for C closed in $X \times X$,
$C[x]$ is closed in X $\forall\, x \in X$.)