Lemma: V symmetric subset of $X \times X$ $\Rightarrow$ $V \circ U \circ V = \bigcup_{(x,y) \in U} V[x] \times V[y]$

Proof: $(x,y) \in$ lhs $\Leftrightarrow \exists\, (x,u) \in V,\ (u,v) \in U,\ (v,y) \in V \overset{\text{symm of } V}{\Longleftrightarrow} (x,y) \in V[u] \times V[v]$ for some $(u,v) \in U$. QED

Uniform topology: Nbhds of x in X are $\{U[x] \mid U \in \mathcal{U}\}$.

$X \times X$ is then given the product topology.

Remark: Every entourage is a nbhd of the diagonal Δ, but not conversely.

Proof: For U^{ent}, choose V^{ent} s/t $\overset{2}{V} \subseteq U$. Replacing V by $V \cap V^{-1}$, may assume V is symmetric. Observe that $V[x] \times V[x]$ is a nbhd of (x,x) that is contained in U. To see that the converse is not true, take $\{(x,y) \in \mathbb{R}^2 \mid |x-y| < 1/(1+x^2)\}$ ☒

Base of a uniformity: A collection of entourages s/t every entourage contains one of ~~these~~ this collection

Characterization of (collections of) subsets that form a base for ~~some~~ uniformities

Axioms to be satisfied for collection $\mathcal{B}$:
- $\Delta \subseteq B \ \forall\, B \in \mathcal{B}$
- Given B_1 & $B_2 \in \mathcal{B}$, $\exists\, B_3 \in \mathcal{B}$ s/t $B_1 \cap B_2 \supseteq B_3$
- $B \in \mathcal{B} \Rightarrow \exists\, B' \in \mathcal{B}$ s/t $B' \subseteq B^{-1}$
- $B \in \mathcal{B} \Rightarrow \exists\, C \in \mathcal{B}$ s/t $C \circ C \subseteq B$
- $\mathcal{B} \neq$ empty.

A collection $\mathcal{B}$ of subsets of $X \times X$ is a base for a uniformity iff these axioms are satisfied.

Subbase for a uniformity: A collection of entourages s/t every entourage contains a finite intersection of elts of this collection.

Sufficient conditions for a collection to be a subbase of some uniformity:

Call the collection $\mathcal{S}$.
- $\Delta \subseteq S \ \forall\, S \in \mathcal{S}$
- $S \in \mathcal{S} \Rightarrow \exists\, T \in \mathcal{S}$ s/t $T \subseteq S^{-1}$
- $S \in \mathcal{S} \Rightarrow \exists\, T \in \mathcal{S}$ s/t $T \circ T \subseteq S$.

In particular, the union of a collection of uniformities is a subbase of a uniformity.

Theorem: X^{uni}; $A \overset{\text{subset}}{\subseteq} X$. $\mathring{A} = \{a \in A \mid U[a] \subseteq A \text{ for some } U \in \mathcal{U}\}$.

Proof: $U[x]$, $U \in \mathcal{U}$, form a nbhd base around x of the uniform topology on X. QED

Theorem: $\mathcal{B}^{\text{base}}$ ($\mathcal{S}^{\text{subbase}}$ of $\mathcal{U}$). For x in X,

$\{B[x] \mid B \in \mathcal{B}\}$ is a base for N_x. $\{S[x] \mid S \in \mathcal{S}\}$ is a subbase for N_x.