Theorem: (Structure theorem for LC-PC spaces). Let X be LC. Then X is PC $\iff$ X is a sum of LC + σ-compact spaces.

(The proof shows: In a LC-PC, we may choose the locally finite open refinement to consist of relatively compact sets; in a LC-σ-compact we may further choose it to be countable.)

Proof: ($\Leftarrow$) Since a sum of PCs is PC, it is ETS LC + σ-compact $\Rightarrow$ PC. Let $(G_\lambda)_{\lambda\in\Lambda}$ be an open cover of X. Let U_n, $n\geq 1$ (with $U_0=\emptyset$) open sets of X s/t $\overline{U}_n \subseteq U_{n+1}$, U_n cover X, U_n relatively compact (by earlier propn; uses LC + σ-compact). Let $K_n := \overline{U}_n \setminus U_{n-1}$. K_n is a compact cover of X. Consider $G_\lambda \cap (U_{n+1}\setminus \overline{U}_{n-2})$, as λ and n vary. These form an open cover of X (being intersection of sets of two open covers: note that $U_{n+1}\setminus\overline{U}_{n-2}$ is a cover of X). For every $x\in K_n$ let W_x be (contained in) one of the sets $G_\lambda\cap(U_{n+1}\setminus\overline{U}_{n-2})$ containing x. By the compactness of K_n, we may choose finitely many W_x that cover K: let's call them $W_{n1},\dots,W_{nk_n}$. By construction the collection $\{W_{n1},\dots,W_{nk_n}\}_n$ is an open refinement of $\{G_\lambda\cap(U_{n+1}\setminus\overline{U}_{n-2})\}$ (and so also of $\{G_\lambda\}$). To prove local finiteness of $\{W_{n1},\dots,W_{nk_n}\}_n$, observe that $U_{n+1}\setminus\overline{U}_{n-2}$ meets $U_{m+1}\setminus\overline{U}_{m-2}$ only if m is in the range $[n+2, n-2]$. Thus $U_{n+1}\setminus\overline{U}_{n-2}$ meets W_{mj} only if m is in the above range, so finitely many W.

($\Rightarrow$) Let V_x be a relatively compact of nbhd of x in X. By PCness, choose $\{U_\alpha\}_I$ a locally finite refinement of $\{V_x\}_x$. Note that U_α are relatively compact. Observation: A compact subset meets only finitely many elts of a locally finite collection.

On the index set I, define an equivalence relation as follows: say $U_\alpha \sim U_{\alpha'}$ if U_α meets $U_{\alpha'}$; take the transitive closure of this relation (which is symmetric & reflexive). Each equivalence class is countable ($\because$ each U_α being relatively compacts meets finitely many of the collection $\{U_\alpha\}$). Write $I=\coprod I_\beta$ where I_β are the equivalence classes. Set $X_\beta := \bigcup_{\alpha\in I_\beta} U_\alpha$. Then $X=\coprod_\beta X_\beta$. Each X_β is LC ($\because X_\beta$ is open in X; and locally closed in LC is LC). Each X_β is σ-compact being a countable union of relatively compact subsets. ▨