I §9. 10. Paracompact spaces

Def: Paracompact = Hdf + every open cover has a locally finite refinement.

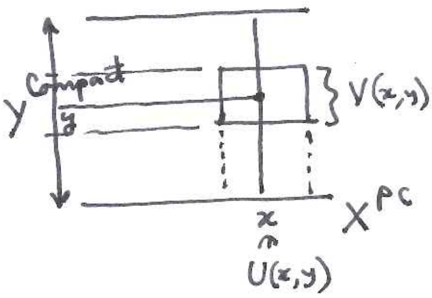
E.g.: Compact ⇒ PC; discrete ⇒ PC (since the open cover consisting of singletons is a locally finite refinement of any open cover).

Direct sum of PCs is PC: If $\{U_\alpha\}$ is an open cover of $\coprod X_i$, then $\{U_\alpha \cap X_i\}_{\alpha i}$ is ~~also~~ a refinement of the original open cover. If X_i is PC, then $\{U_\alpha \cap X_i\}_\alpha$ admits a locally finite refinement ($\forall i$). The union over i of such refinements gives a locally finite refinement of the original cover.

- Open subspace of a compact space need not be paracompact.
- Product of two PCs need not be PC.

Propn: Closed subspace of a PC is PC. **Proof:** Easy.

Propn: Compact × PC = PC. **Proof:** X^{PC}, Y^{compact}. $X \times Y$ Hdf OK.



Given an open cover of $X \times Y$, we choose for each $(x,y) \in X \times Y$ a basic open set $U(x,y) \times V(x,y)$ with $U(x,y)$ open in X, $V(x,y)$ open in Y, $(x,y) \in U(x,y) \times V(x,y)$, and s/t $U(x,y) \times W(x,y)$ is contained in an open set of the cover. Fixing x and varying y, using QCness of Y, there exist finitely many $y_1, \dots, y_n$ s/t $V(x,y_1) \cup \dots \cup V(x,y_n) = Y$. Set $U(x) :=$ intersection of $U(x,y_1) \cap \dots \cap U(x,y_n)$. Then $U(x)$ is open around x, and

$U(x) \times V(x,y_j) \subseteq U(x,y) \times V(x,y_j)$.

Now let $\{U_\alpha\}$ be a locally finite refinement (in X) of the open cover $\{U(x) \mid x \in X\}$. For each α, choose x_α in X such that $U_\alpha \subseteq U(x_\alpha)$. ~~Now $\{U_\alpha \times V(x_\alpha, y$~~ Let $y_1, \dots, y_n$ be the elements of Y associated to x_α as in the previous paragraph. Then CLAIM $R := \{U_\alpha \times V(x_\alpha, y_1), \dots, U_\alpha \times V(x_\alpha, y_n)\}_\alpha$ is an open cover of $X \times Y$ and is a locally finite refinement of $\{U(x,y) \times V(x,y)\}_{(x,y)}$.

The part about refinement is clear from the construction. Since U_α is locally finite in X, given $x \in X$ $\exists$ U open around x s/t U meets only finite many U_α. Now $U \times Y$ meets only finitely many members of R.

Finally the elements of R cover $X \times Y$. Given (x,y), choose α s/t $x \in U_\alpha$ and then $y_j \in Y$ s/t $y \in V(x_\alpha, y_j)$ [this is possible since $V(x_\alpha, y_1) \cup \dots \cup V(x_\alpha, y_n) = Y$]. QED