I§9. No. 9. Locally compact & $\sigma$-compact spaces 

An LC space is <u>countable at infinity</u> or <u>$\sigma$-compact</u> if it is a countable union of compact subsets.

E.g. A discrete space is $\sigma$-compact iff it is countable; $\mathbb{R}^n$;

It is possible for a Hausdorff space to be $\sigma$-compact without being LC (Hilbert space with weak topology).

<u>Propn:</u> $X$ LC, $\sigma$-compact. Then $\exists$ sequence ~~of~~ $U_1, U_2, \ldots$ of relatively compact open sets, whose union is all of $X$, and s/t $\overline{U_{n-1}} \subseteq U_n \ \forall\ n \geq 1$.

<u>Proof:</u> Write $X = \bigcup_{i=1}^{\infty} K_i$, $K_i$ compact. By LC (see earlier propn) $K_1$ has a relatively compact open nbhd $U_1$ (in fact, the relatively compact open nbhds of any compact set form a FSN). Now, inductively choose $U_n$ to be a relatively compact open nbhd of $\overline{U}_{n-1} \cup K_n$. QED.

<u>CoR:</u> With notation as in the proposition, each compact set is contained in some $U_n$.

<u>CoR:</u> $X^{LC}$, $\tilde{X}$ = One-point compactification. $X$ is $\sigma$-compact $\Leftrightarrow$ There is a countable ~~fundamental sy~~ FSN at infinity of $\tilde{X}$.

<u>Proof:</u> $\Leftarrow$ $\mathcal{B} = \{B_1, B_2, \ldots\}$ be an FSN at $\infty$. ~~Write~~ Write $\tilde{X} \setminus B_i = X \setminus K_i$. Then $K_i$ compact in $X$ (by the defn of $\tilde{X}$) and $\bigcup K_i = X$ (since $\bigcap B_i = \{\infty\}$ by accessibility of $\tilde{X}$).

$\Rightarrow$ Choose $U_i$ as in Propn. ~~For~~ Claim that $\tilde{X} \setminus \overline{U}_n$ is a FSN at $\infty$: Given $K^{\text{compact}}$ in $X$, we have $K \subseteq U_n \subseteq \overline{U}_n$ for some $n$, so $\tilde{X} \setminus \overline{U}_n \subseteq \tilde{X} \setminus K$. QED

<u>Passage of the LC + $\sigma$-compact property:</u> • Every closed subspace of a $\sigma$-compact + LC space is $\sigma$-compact + LC.

- However open sets of compact sets need not be $\sigma$-compact.
  E.g. take the Alexandroff compactification of a LC but not $\sigma$-compact space.
- Finite products of LC + $\sigma$-compact spaces are LC + $\sigma$-compact.