I §9. No. 7 Locally compact spaces.

__Def:__ Top. space X is __locally compact__ if X is Hdf and every ~~set~~ point of X has a compact nbhd.

__Ex:__ Compact $\Rightarrow$ Locally compact. Discrete spaces are locally compact but not compact except if finite. $\mathbb{R}$ is locally compact but not compact

__Propn:__ ~~Every~~ locally compact $\Rightarrow$ regular.

__Proof:__ Recall the following: if every point has a closed regular nbhd, then the space is regular. (We can verify (O_{III}) in this case.) But a compact nbhd is both closed and regular (compact $\Rightarrow$ regular as we've seen). ▨

__Propn:__ In a locally compact space every point has a FSN of compact nbhds.

__Proof:__ Being regular, ~~we have~~ the closed nbhds form a FSN. Intersecting the closed nbhds with a given compact nbhd, ~~gives~~ we obtain a compact FSN. ▨

Z: $\exists$ non-Hdf spaces in which every point has a FS of compact nbhds.
E.g. ~~[illegible]~~ See Example 1 on page 40 of the notes.

! ~~The~~ __Generalizing the last proposition__: every compact set K of a locally compact space has a FS of compact nbhds. __Proof:__ Let U ~~open~~ nbhd $\supseteq K$. For $k \in K$, $\exists$ a compact nbhd C of k s/t $k \in C \subseteq U$. As k varies over K, the ~~sets~~ C cover K. By quasi-compactness of K (and since C is a nbhd of k), $\exists$ finitely many C that cover K. The union of these is a compact nbhd of K contained in U. ▨

__Propn:__ X locally compact. $F \subseteq X$. If $F \cap K$ is closed in K $\forall$ K compact $\subseteq X$, then F is closed.

__Proof:__ As already seen if $F \cap S$ is closed in S for a collection S of subsets of X s/t the interiors of S cover X, then F is closed. In a locally compact space, since every point has a compact nbhd, the ~~compact sets have~~ interiors of compact sets cover X. QED

In analogy with compact in Hdf is closed: locally compact in Hdf is locally closed.
and with closed in compact is compact: locally closed in locally compact is locally compact.

__Products__: If X_i are locally compact and compact except for finitely many i then $\prod X_i$ is LC.
(Pf: ETS LC$\times$LC is LC by Tychonoff. This is also easily checked.)

• If $\prod X_i$ is locally compact, and none of the X_i is empty, then the X_i are locally compact and moreover almost all X_i are compact. (Proof: Choose a compact nbhd, say V, of a point in $\prod X_i$. Now $p_i(V) = X_i$ for almost all i, so X_i is quasi-compact except for finitely many i. Each X_i is homeo to a closed subspace of $\prod X_i$, and so is locally compact.)