I §9 No. 6 Inverse limits of compact spaces

Proposition: I directed set. $(X_\alpha, f_{\alpha\beta})$ inverse system (of top spaces & cont. maps) indexed by I.
Let $X = \varprojlim X_\alpha$ be the inverse limit, $f_\alpha : X \to X_\alpha$ canonical map (continuous).
Assume that the X_α are compact. Then ① X is compact & $\begin{cases} f_\alpha(X) = \bigcap_{\beta \geq \alpha} f_{\alpha\beta}(X_\beta) \\ \forall \alpha \in I \end{cases}$
and ② If the X_α are non-empty then so is X.

Proof: X is closed in $\prod X_\alpha$ (it can be realized, as already seen, as ~~the~~ intersections of pullbacks of diagonals under appropriate continuous maps). Thus X is compact.
As to the rest, we apply the following theorem ~~from~~ about inverse limits of sets.
We take $\mathfrak{S}_\alpha$ to be the collection of closed sets of X_α. Condition ⓘ clearly holds.
ⓘⓘ holds ∵ X_α are QC by hypothesis ⓘⓘⓘ holds because singletons are closed ((H^1))
and $f_{\alpha\beta}$ are continuous. ⓘⓥ holds ∵ M_β is QC and so $f_\alpha(M_\beta)$ is QC and also closed since X_α is Hdf. QED.

Theorem (about inverse limits, from Set Theory Chap III, §7, no. 4, Theorem 1)
criterion for non-emptiness

I directed set. $(X_\alpha, f_{\alpha\beta})$ inverse system indexed by I. Suppose that
$\forall \alpha \in I$ we are given a collection $\mathfrak{S}_\alpha$ of subsets of X_α s/t:

ⓘ $\mathfrak{S}_\alpha$ is closed under arbitrary intersections (in particular, $X_\alpha \in \mathfrak{S}_\alpha$)

ⓘⓘ $\mathfrak{S}_\alpha$ satisfies FIP: if the intersection of a family of elts of $\mathfrak{S}_\alpha$ is empty, then the intersection of some finitely many elts of that family is also empty.

ⓘⓘⓘ $\forall$ pair $\alpha \leq \beta$, $\forall x_\alpha \in X_\alpha$, we have $f_{\alpha\beta}^{-1}(x_\alpha) \in \mathfrak{S}_\beta$.

and ⓘⓥ $\forall$ pair $\alpha \leq \beta$, $\forall M_\beta \in \mathfrak{S}_\beta$, we have $f_{\alpha\beta}(M_\beta) \in \mathfrak{S}_\alpha$.

Then ① $\forall \alpha \in I$, $f_\alpha(\varprojlim X_\alpha) = \bigcap_{\beta \geq \alpha} f_{\alpha\beta}(X_\beta)$.

and ② $\varprojlim X_\alpha$ non-empty if none of the X_α is empty.