# I §9 No. 4. Image of a compact space under a continuous mapping

Theorem: Image of a QC space under a continuous map is QC. Pf: Verify QC$^{\text{III}}$ Borel-Lebesgue.

CoR: $f \overset{\text{cont}}{:} X \to X'^{\,\text{Hdf}}$. The image under $f$ of any QC/rel. ~~compact~~ Quasi-compact is compact/rel compact

CoR: $f \overset{\text{cont}}{:} X^{QC} \to Y^{\text{Hdf}}$. Then $f$ is closed. If $f$ is bijective then it's a homeo.

CoR: A Hdf topology that is coarser than a QC topology equals the latter.

CoR: Let $R$ be a Hdf equivalence relation on $X$. ⓐ If $\exists K \overset{QC}{\subseteq} X$ s/t $K$ meets every $R$-equiv class, then $X/R$ is compact & the canonical mapping $K/R_K \to X/R$ is a homeo. ⓑ If $K$ meets every equiv. class precisely once, then $K$ is a continuous section of $X$ w.r.t. the relation $R$.

# I §9 No. 5 Product of Compact Spaces

Theorem (Tychonoff) Every product of QC/Compact spaces is QC/compact. Conversely if a product of non-empty spaces QC/Compact, then each factor is so.

Proof: ETS QC part ($\because$ Hdf part shown earlier) If $\prod X_i$ is QC and none of the $X_i$ is empty, then since $\prod X_i \xrightarrow{p_i} X_i$ is onto and continuous, $X_i$ is QC. Now suppose each $X_i$ is QC. Let $\mathcal{U}$ be an ultrafilter on $\prod X_i$. Then $p_i(\mathcal{U})$ is an ultrafilter on $X_i$ (we assume that no $X_i$ is empty, for otherwise $\prod X_i$ is empty, so the conclusion holds trivially). Since $X_i$ is QC $p_i(\mathcal{U}) \to x_i \in X_i$. This means $\mathcal{U} \to (x_i)$ [see I §7 No. 6]. QED

CoR: For $A \overset{\text{subset}}{\subseteq} \prod X_i$ to be relatively Quasi compact, it is necessary and sufficient that $p_i(A)$ be relatively Quasi compact in $X_i$ for every $i$.

Proof: Necessity: $\because$ rel. Quasi compact is preserved by the projection $p_i$, since $p_i$ is continuous. Sufficiency: Tychonoff. ☒