I §9 Compact spaces & locally compact spaces No. 1 Quasi-compact Spaces & Compact spaces (38)

Def: Axiom for quasi-compactness: (QC) $\boxed{\text{every filter admits at least one cluster point}}$

Compact := Quasi-compact + Hdf

Three other equivalent axioms: (QC') Every ultrafilter is convergent

(QC'') If the intersection of a family of closed sets is empty, then the intersection of some finitely many in the family is also empty.

(QC''') (Borel-Lebesgue) Every open cover has a finite subcover.

Rmk: $Z \xrightarrow{f} X$ quasi-compact (f set map) $\mathcal{U}$ ultrafilter on Z. ~~Then f(U) has~~ Then f has at least one limit point w.r.t. $\mathcal{U}$.

Examples: • Finite sets are QC. More generally, a top space with finitely many open sets is QC. A finite space is compact iff it is Hdf iff it is discrete. Conversely, a discrete compact space is finite.

• The finite complement top is QC.

• The notion of QC is mainly useful in algebraic geometry. In other theories it is compactness that is useful.

Theorem: Let X QC, $\mathcal{F}$ a filter on X. ~~AAAA~~ Then $\mathcal{F}$ contains every nbhd of the set of its cluster points.

Proof: ETS that every $\mathcal{U}$ ultra finer than $\mathcal{F}$ contains every such nbhd say N. By QC hypothesis, ~~∃~~ $\mathcal{U} \to x$ for some x. Since $\mathcal{U} \geq \mathcal{F}$, the point x is a limit point of $\mathcal{F}$.

Thus N is a nbhd of x and so belongs to $\mathcal{U}$. QED

CoR: For a filter on a compact set to converge it is necessary and sufficient that it has a single cluster point. Proof: $\Rightarrow$ Hdf axiom (H^5).

$\Leftarrow$ By the Theorem, the filter is finer than the nbhd filter of the point. QED