I $\S 8$. Hausdorff & Regular Spaces. No. 6 <u>Equivalence Relations on a regular space</u>.

<u>Propn:</u> Let $X \xrightarrow{\pi} X/R$ be a quotient map, where R is "pinching a closed set to a point". Assume that X is REGULAR. Then X/R is Hdf (but not in general regular).

<u>Proof:</u> For two points in the quotient, neither of which is the pinched one, ~~we only~~ we just take nbhds that are disjoint around the corresponding points in X and further not meeting F (where F is the closed set that is pinched) (possible since $X \setminus F$ is open in X and Hdf). These are saturated.

Suppose one of the points is the pinched one. Then we separate, by disjoint ~~closed sets~~ nbhds, the closed set F and the other point. These are saturated. QED

<u>RESULT:</u> If R is an open & closed relation on X REGULAR, Then X/R is Hdf (Criterion for Hdfness of X/R).

<u>Proof:</u> $X \times X \xrightarrow{\text{quotient}} \dfrac{X \times X}{R \times R} \to \dfrac{X}{R} \times \dfrac{X}{R}$ continuous bijection (map exists by universal property of the quotient by $R \times R$)

The pull-back of the diagonal of X/R to $X \times X$ is the graph $G := \{(a,b) \in X \times X \mid a \underset{R}{\sim} b\}$ of the relation R. Thus G being closed is a necessary condition for X/R to be Hdf.

~~Suppose~~ Under the hypothesis that X is regular and R is closed, G is closed (see Proposition below). Since R is open, the map $X \times X \to X/R \times X/R$ is open (product of two open maps is open) and so $\dfrac{X \times X}{R \times R} \overset{\text{homeo}}{\simeq} \dfrac{X}{R} \times \dfrac{X}{R}$.

Observe that G is saturated for $R \times R$. So the image of G in $X \times X / R \times R$ is closed. In $X/R \times X/R$, the image of G is closed and this is the diagonal of X/R. ☒

<u>PROPN:</u> (CRUCIAL INGREDIENT IN THE RESULT ABOVE). X REGULAR and R CLOSED on X. Then the graph G of R in $X \times X$ is closed.

<u>Proof:</u> Suppose $(a,b) \notin G$. ETS that $\exists$ nbhds U, V around a and b that are disjoint and one of which is saturated (so that $(U \times V) \cap G =$ empty). Since $(a,b) \notin G$, we have $\{a\}^{sat} \cap \{b\} =$ empty. Note $\{a\}$ is closed (X is Hdf so accessible) and so $\{a\}^{sat}$ is closed (since R is closed).

Thus $X \setminus \{a\}^{sat}$ is a saturated open nbhd of b. By regularity (closed nbhds form an FSN around any point) $\exists\ C$ closed nbhd of b s/t $C \subseteq X \setminus \{a\}^{sat}$. Since $X \setminus \{a\}^{sat}$ is saturated, $C^{\text{saturated}} \subseteq X \setminus \{a\}^{sat}$. Since R is closed, $C^{\text{saturated}}$ is closed. The nbhds $X \setminus C^{\text{saturated}}$ of a and $C^{\text{saturated}}$ of b (or even C of b) have the desired properties. ☒