I §8 Hdf Spaces & Regular Spaces **No. 5 Extension by Continuity; Double limit**

Theorem: Let X be a top space, $A \overset{\text{dense}}{\subseteq} X$. Let $f: A \to Y^{\text{top sp}}$ be a set map. Suppose that the following holds: $\lim_{z \to x,\, z\in A} f(z)$ exists $\forall x \in X$.
Then if Y is Hdf the limit is unique and we may define $f(x)$ to be the above limiting value.
If Y is regular, then the $f: X \to Y$ so defined is continuous.

Remark: Regularity of Y cannot be just dropped without making other changes.

Proof: We have to show $\lim_{z\to x} f(z) = f(x)$ $\forall x \in X$. In other words, we're to show that $\forall$ nbhd V of $f(x)$ there exists open nbhd U of x s/t $fU \subseteq V$. By regularity of Y, we may assume V is closed, for closed nbhds form an FSN around every point. By the hypothesis that $\lim_{z\to x,\, z\in A} f(z) = f(x)$, it follows that $\exists$ a nbhd U of x s/t $f(U\cap A) \subseteq V$. Since V is closed, we have $\overline{f(U\cap A)} \subseteq V$. Thus it is enough to prove the claim that: $fU \subseteq \overline{f(U\cap A)}$. Proof of claim: take $u \in U$. N_u leaves a trace $U\cap A$ since $U \cap A$ is dense in U ($\because U$ open) and so $\lim_{z\to u,\, z\in U\cap A} f(z) = f(u)$ [since by hypothesis $\lim_{z\to u,\, z\in A} f(z) = f(u)$]. Thus fu is a cluster point of $N_u \cap (U\cap A)$. In particular it belongs to the closure of $X \cap (U \cap A) = U\cap A$. QED.

The topology on $A = X_1 \times X_2$ is discrete, so any function on A is continuous and limits exist when we approach points of A.

COROLLARY (Double limit). Let $X = X_1 \times X_2$ where X_1, X_2 are sets. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be filters on X_1, X_2. Let f be a mapping of X into a regular top space Y. Suppose that ① $\lim_{\mathfrak{F}_1 \times \mathfrak{F}_2} f$ exists and ② $\lim_{x_2,\, \mathfrak{F}_2} f(x_1, x_2) = g(x_1)$ exists $\forall x_1 \in X_1$.
Then $\lim_{x_1,\, \mathfrak{F}_1} g(x_1)$ exists and equals $\lim_{\mathfrak{F}_1, \mathfrak{F}_2} f$.

Proof: Apply theorem above as follows. Make $\mathfrak{F}_1, \mathfrak{F}_2$ to be nbhd filters on $X_1' = X_1 \amalg \{\omega_1\}$ and $X_2' = X_2 \amalg \{\omega_2\}$ as already seen. Take X in the theorem to be $(X_1 \times X_2') \amalg \{(\omega_1, \omega_2)\}$. Take A to be $X_1 \times X_2$. Hypothesis ① implies $\lim_{(x_1,x_2)\to(\omega_1,\omega_2),\, (x_1,x_2)\in A} f(x_1,x_2)$ exists.
② implies $\lim_{(x_1,x_2)\to(x_1,\omega_2),\, (x_1,x_2)\in A} f(x_1,x_2)$ exists ~~(note $\{x_1\}\times X_2$ is open in $X_1 \times X_2'$)~~
(note $\{x_1\} \times X_2$ is open in A, so whether we restrict to A or to $\{x_1\}\times X_2$ while approaching (x_1, ω_2), it does not matter). Thus the hypothesis of the Theorem holds.
By the conclusion $\lim_{(x_1,x_2)\to(\omega_1,\omega_2)} f(x_1,x_2)$ exists where (x_1,x_2) varies over $(X_1 \times X_2') \amalg \{(\omega_1,\omega_2)\}$. If we approach (ω_1, ω_2) by a more restrictive path, then the limit exists (even better) ~~If we restrict to A we get lim~~ and all such limits are equal. If we restrict ourselves to A, we get $\lim_{\mathfrak{F}_1 \times \mathfrak{F}_2} f$.
If we restrict ourselves to $X_1 \times \{\omega_2\}$ we get $\lim_{x_1,\, \mathfrak{F}_1} g(x_1)$. QED