I§8 Hausdorff spaces & Regular spaces. No 1. Hausdorff spaces.

<u>Propn</u> TFAE for X^{top} space: (H) (Hausdorff axiom) two distinct points have disjoint nbhds (H^1) The intersection of closed nbhds of any point consists only of that point (H^2) The diagonal in $X \times X$ is closed.

(H^3) For any index set I, the diagonal in X^I is closed

(H^4) No filter admits more than one limit point.

(H^5) If a filter has a ~~cluster~~ limit point, then that points is ~~a~~ ~~limit~~ The only cluster point.

<u>Proof scheme</u>: $H \Rightarrow H^1 \Rightarrow H^5 \Rightarrow H^4 \Rightarrow H$; $H \Rightarrow H^3 \Rightarrow H^2 \Rightarrow H$.

<u>Def</u>: if X satisfies the conditions of the propn, it is <u>Hausdorff</u>

<u>Propn</u>: $f, g^{\text{continuous both}} : X \to Y^{\text{Hausdorff}}$; Then $\{x \mid fx = gx\}$ is closed in X.

! <u>Proof</u>: $\{x \mid fx = gx\}$ is the inverse image of the diagonal of Y under $X \xrightarrow{(f,g)} Y \times Y$. ▨

<u>CoR</u>: $f, g^{\text{both continuous}} : X \to Y^{\text{Hausdorff}}$; if f and g agree on a dense set of X, Then $f = g$.

(PRINCIPLE OF EXTENSION OF IDENTITIES)

<u>CoR</u>: $f^{\text{cont}} : X \to Y^{\text{Hausdorff}}$; then the graph of f (i.e., $\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$) is closed.

! <u>Proof</u>: The graph is the inverse image of the diagonal under $X \times Y \xrightarrow{(f, id)} Y \times Y$. ▨

<u>Propn</u>: Given a finite subset of X, ~~each~~ there exist nbhds of every point of the subset that are mutually disjoint.

<u>CoR</u>: Hausdorff & finite $\Rightarrow$ discrete

<u>CoR</u>: Finite sets of Hausdorff spaces are closed. <u>Proof</u>: $\because$ singletons are closed by (H^2). ▨

<u>Propn</u>: If for every pair $x \neq x'$ of points of a top space X, there exists ~~a~~ continuous mapping f from X to a Hausdorff space Y s/t $fx \neq fx'$ then X is Hausdorff.

<u>Proof</u>: Pull back disjoint nbhds of fx & fx' under f. ▨

<u>CoR</u>: Any topology finer than a Hausdorff topology is Hausdorff.

<u>CoR</u>: Subspaces of Hausdorff spaces are Hausdorff.