I § 7. Limits No. 4 (Continued) Limits & Continuity

Remark: $X \xrightarrow{f} Y$ ~~top spaces~~ map of top. spaces continuous at a in X

Let $Z \xrightarrow[\text{set}]{g} X$, and $\mathcal{F}$ a filter on Z such that $\lim_{\mathcal{F}} g = a$.

Then $\lim_{\mathcal{F}} f \circ g = f(a)$.

I § 7 No. 5 Limits relative to a subspace

Let $A \subseteq X$ top space. Let $a \in \overline{A}$. Let $f : A \to Y$ top space be a map.

We write $\lim_{x \to a,\, x \in A} f(x) = y$ if the direct image filter under f of the trace of the nbhd filter $\mathcal{N}_a$ of a in A is finer than the nbhd filter $\mathcal{N}_y$.

In the case when a is not an isolated point & $A = X \setminus \{a\}$, we write $\lim_{\substack{x \to a \\ x \neq a}} f(x)$ for $\lim_{x \to a,\, x \in A} f(x)$. For a and A like this, a map $f : X \to Y$ is continuous at a iff $\lim_{x \to a,\, x \neq a} f(x) = f(a)$.

Now consider $B \subseteq A$ and suppose that $a \in \overline{B}$. Let $f : A \to Y$ top space be a map. Then if $y = \lim_{\substack{x \to a \\ a \in A}} f(x)$, then $y = \lim_{\substack{x \to a \\ a \in B}} f(x)$. The converse is not in general true, but if $B = V \cap A$ where V is a nbhd in X of a, then it holds.

I § 7 Limits. No. 6 Limits in initial topologies: application to products limits in quotients.

Propn: X^{set}. Let X_i be top spaces $(i \in I)$ & $f_i : X \to X_i$ set maps. Give X the initial top w.r.t f_i. Let $\mathcal{F}$ be a filter on X and let $x \in X$. Then $\mathcal{F} \to x$ iff $f_i(\mathcal{F}) \to f_i x \ \forall i$. Proof: $\Rightarrow$ by continuity of f_i. $\Leftarrow$ If $F_i \in \mathcal{F}$ is s/t $f_i F_i \subseteq U_i$ for U_i nbhd of $f_i x$ in X_i, for $1 \le i \le n$, then $F_1 \cap \dots \cap F_n \subseteq f_1^{-1}(U_1) \cap \dots \cap f_n^{-1}(U_n)$. QED

COR: Let $\mathcal{F}$ be a filter on $\prod X_i$. Then $\mathcal{F} \to (x_i)$ iff $p_i(\mathcal{F}) \to x_i \ \forall i$.

COR: Let $f_i : X \to X_i$ be maps of a set to top spaces X_i. Let $\mathcal{F}$ be a filter on X. Then $\lim_{\mathcal{F}} \prod f_i = (x_i)$ iff $\lim_{\mathcal{F}} f_i = x_i \ \forall i$.

Propn: Let $X \xrightarrow{\pi} X/R$ be an open quotient map. Let $x \in X$ and $\mathcal{B}'$ a filter base on X/R s/t $\mathcal{B}' \to \pi x$. Then $\exists$ filter base $\mathcal{B}$ on X s/t $\mathcal{B} \to x$ & $\pi\mathcal{B}$ is equivalent to $\mathcal{B}'$.

Proof: The hypothesis of openness implies that x is a cluster point of $\pi^{-1}\mathcal{B}'$. Let $\mathcal{B}$ be the filter base consisting of elts of the form $\pi^{-1}B' \cap N$, with $B' \in \mathcal{B}'$ & N a nbhd of x. Then $\mathcal{B}$ has the desired property. QED