I $\S_S$ 7 Limits No. 3 (Continued) Limits & Cluster values of a function

Example: (Nets) Let X be a directed set & $f: X \to Y^{\text{top space}}$ a map. Limits and cluster values of f are defined (w.r.t. the section filter on X). Explicitly: $y = \lim f$ iff given $V^{\text{nbhd of } y}$ $\exists\, x_0 \in X$ s/t $f^{-1}(V) \supseteq S(x_0)$ (where, recall $S(x_0) := \{x \in X \mid x \geq x_0\}$); y is a cluster value of f iff every nbhd of V of y meets $f(S(x))$ for every x in X (in other words, ~~every nbhd~~ the inverse image $f^{-1}(V)$ and the section $S(x)$ meet $\forall\, V^{\text{nbhd of } y}$ & $\forall\, x \in X$).

Recall our ~~set~~ notation: $X^{\text{set}} \xrightarrow{f} Y^{\text{top sp}}$, $\mathfrak{F}$ filter on X.

Remark: • The coarser the topology on Y, the more the limits and cluster values. The finer the filter $\mathfrak{F}$, the more the limits but fewer the cluster values.

• The set of cluster values of f (w.r.t $\mathfrak{F}$) is closed (possibly empty).

Propn: Limits and cluster values of f w.r.t. $\mathfrak{F}$ depend only upon the germ of f w.r.t. $\mathfrak{F}$. More precisely if $g: X \to Y$ is s/t $\tilde{g} = \tilde{f}$ then the limits (respectively cluster values) of f coincide with those of g.

I $\S_S$ 7 Limits No. 4 Limits & Continuity

Now let $X \xrightarrow{f} Y$ be a map of top spaces. If $\mathfrak{F}$ is the ~~[illegible]~~ nbhd filter of a point a of X, we write $\lim_{x \to a} f(x) = y$ in place of $\lim_{x, \mathfrak{F}} f(x) = y$. We also use the terms y is a limit of f at a, y is a cluster value of f at a, where "at a" ~~[illegible]~~ is to be understood ~~at~~ as "w.r.t. the nbhd filter $\mathfrak{F}$ at a".

Propn: f continuous at a iff $\lim_{x \to a} f(x) = f(a)$. **Proof:** From the defn, f is continuous at a iff the direct image of the nbhd filter at a is finer than the nbhd filter of $f(a)$.

Remark: The continuity of f at a depends only upon the germ of f at a.

Propn: If f is cont at a, then given a filter base $\mathfrak{B}$ converging to a, the direct image $f(\mathfrak{B})$ converges to $f(a)$. Conversely, if $\forall$ ultrafilter $\mathcal{U}$ converging to a ~~[illegible]~~ the direct image $f(\mathcal{U})$ converges to $f(a)$, then f is continuous at a. **Proof** (of the second assertion) By contradiction, suppose $\exists\, V$ nbhd of $f(a)$ s/t $f^{-1}(V)$ is not a nbhd of a. Let $\mathcal{U}$ be an ultrafilter in X finer than the nbhd filter of a but not containing $f^{-1}(V)$.

Claim: $f(\mathcal{U})$ does not converge to $f(a)$. In fact, $\nexists\, U$ in $\mathcal{U}$ s/t ~~[illegible]~~ $fU \subseteq V$. $\because$ if this held, then $U \subseteq f^{-1}(V)$, but this is a $\Rightarrow\Leftarrow$ since $f^{-1}(V) \notin \mathcal{U}$. QED