I §7/5 ~~Filters~~ No. 2. Cluster point of a filter base. A point $x \in X^{top}$ is a cluster point of a filter base $\mathfrak{B}$, if x is in the closure of every elt of $\mathfrak{B}$.

If $\mathfrak{B}'$ is ~~finer~~ coarser than $\mathfrak{B}$ then a cluster point of $\mathfrak{B}$ is also one of $\mathfrak{B}'$.

Thus cluster points depend only on the filter gen by $\mathfrak{B}$ and not so much on $\mathfrak{B}$.

Remarks:
- x is a cluster point of $\mathfrak{B}$ iff every set in a FSN of x meets every $B \in \mathfrak{B}$.
- x is a cluster point of ~~$\mathfrak{B}$ iff~~ a filter $\mathfrak{F}$ iff $\exists$ filter finer than $\mathfrak{F}$ & the nbhd filter at x.
- every limit point is a cluster point.
- cluster points of an ultra filter are limit points.

Propn: The set of cluster points of a filter base is closed ($\because$ it is $\bigcap_{B \in \mathfrak{B}} \overline{B}$).

Suppose $A \subseteq X$. If $\mathfrak{B}$ is a filter base consisting of subsets of A, then any cluster point in X of $\mathfrak{B}$ belongs to $\overline{A}$. Conversely, ~~for $x \in \overline{A}$~~ every x in $\overline{A}$ is the limit point of a filter base consisting of subsets of A: indeed, ~~for x~~ define $\mathfrak{B}$ to be the trace in A of the nbhd filter of x; then $\mathfrak{B}$ is in fact a filter on A and it generates in X a filter finer than the nbhd filter of x.

I §7 Limits No. 3 Limiting values (or limits) & Cluster values of a function.

Let $f : X \to Y^{\text{top space}}$ and $\mathfrak{F}$ a filter on X. We say that $y \in Y$ is ~~the~~ a limiting value (or limit) of f (w.r.t. the filter $\mathfrak{F}$) if the filter base $f(\mathfrak{F})$ converges to y.

We say y is a cluster value of f (w.r.t. $\mathfrak{F}$) if y is a cluster point of $f(\mathfrak{F})$.

When y is ~~the~~ a limit of f we write: $\lim_{\mathfrak{F}} f = y$, $\lim_{x, \mathfrak{F}} f(x) = y$, or $\lim_{x} f(x) = y$

Propn: y in Y is ~~the~~ limit of f (w.r.t. $\mathfrak{F}$) iff $\forall\, V^{\text{nbhd}}$ of y $\exists\, F \in \mathfrak{F}$ s/t $f(F) \subseteq V$ (in other words, iff $f^{-1}(V) \in \mathfrak{F}$ $\forall\, V^{\text{nbhd}}$ of y).

$\left\{ \right.$ y in Y is a cluster value of f $\iff$ f^{-1} of the nbhd filter of y is defined & $\exists$ filter in X finer than $\mathfrak{F}$ & f^{-1}(nbhd filter of y).

Examples: • Think (both) of a sequence $\{y_n\}$ in a top space Y as a map $\mathbb{N} \xrightarrow{\varphi} Y$ ~~from $\mathbb{N}$ to Y~~. The notion of limits & cluster values of a sequence are recovered by the corresponding notions for the pair $(\mathfrak{F}, \varphi)$ where $\mathfrak{F}$ is the Fréchet filter on $\mathbb{N}$.

~~Limits and~~ Cluster values of a sequence are to be distinguished from cluster points of the underlying set of the sequence. While any cluster value of a sequence is a cluster point of the set, the converse is not true in general.

$\iff \exists$ filter $\mathfrak{F}'$ in X finer than $\mathfrak{F}$ and s/t y is a limit of f w.r.t $\mathfrak{F}'$.