 <u>Germs with respect to a filter.</u>

Let $\mathfrak{F}$ be a filter. On $\mathcal{P}(X)$ define equivalence relation: $U \sim V$ if $\exists\, F \in \mathfrak{F}$ s/t $U \cap F = V \cap F$. The class of U is the <u>germ of U</u> w.r.t. $\mathfrak{F}$. The operations of union & intersection pass to germs. ~~With respect to these~~ Each of these laws is commutative & associative. Every element is an idempotent with respect to either law. Each law distributes over the other.

For ~~germs~~ germs α and β, we have $\alpha \cap \beta = \alpha \iff \alpha \cup \beta = \beta$. We write $\alpha \subseteq \beta$ if these hold. The relation $\subseteq$ is an ordering w.r.t. which the germs form a lattice. The germ of the empty set is the least germ and the germ of X the largest germ.

<u>Germs of functions</u> (w.r.t. $\mathfrak{F}$) Let Y be a set. Let Φ be the set of pairs (F, f) where $F \in \mathfrak{F}$ and $f : F \to Y$ is a set map. Define equivalence relation $\sim$ on Φ: $(F, f) \sim (F', f')$ if $\exists\, G \in \mathfrak{F}$ s/t $G \subseteq F \cap F'$ and $f|_G = f'|_G$. $\Phi/\sim$ is the set of <u>germs of mappings</u> of X to Y (w.r.t. $\mathfrak{F}$). Note:

- Every element (F, f) of Φ is equivalent to an element of the form (X, g): define g to be f on F and arbitrarily outside of F.
- For M & N in $\mathcal{P}(X)$ the characteristic fns of M and N have the same germ if M & N have the same germ.

If $Y \xrightarrow{\varphi} Z$ is a set map, then (post) composing with φ induces a map from {germs of maps to Y} to {germs of maps to Z}.

<u>Finite product of germs</u> Let $Y = \prod_{i=1}^{n} Y_i$ finite product. Let $\widetilde{\Phi}_i$ be the set of germs of maps from X to Y_i; let $\widetilde{\Phi}$ be the set of germs of maps from X to Y. Then we have a bijection $\prod_{i=1}^{n} \widetilde{\Phi}_i \to \widetilde{\Phi}$ given by $((F_1, f_1), \dots, (F_n, f_n)) \mapsto (F_1 \cap \dots \cap F_n, (f_1, \dots, f_n))$

<u>Germs of maps to a ~~groups~~ group/ring/algebra has the same structure</u> Suppose Y is a set with a binary operation $Y \times Y \xrightarrow{\mu} Y$ (law of composition). Then the set $\widetilde{\Phi}$ of germs of maps from X to Y inherits a law of composition $\widetilde{\mu}$:

$$\widetilde{\Phi} \times \widetilde{\Phi} \underset{\text{bijection}}{\cong} \text{set of germs from } X \text{ to } Y \times Y \xrightarrow[\text{(Post) Compose with } \mu]{} \widetilde{\Phi}.$$

μ commutative / associative / has identity $\Rightarrow$ ditto for $\widetilde{\mu}$

Y is a group/ring/algebra $\Rightarrow$ ditto for $\widetilde{\Phi}$.