I §6 Filters No. 7 Product of filters. Let $\mathfrak{F}_i$ be filters respectively on sets X_i. Consider $\{\prod F_i \mid F_i \in \mathfrak{F}_i,\ F_i = X_i$ for almost all $i\}$ is a filter base on $\prod X_i$. (In fact, if $\mathcal{B}_i$ is a base of $\mathfrak{F}_i$ then the collection $\{\prod F_i \mid F_i = X_i$ for almost all i, $F_i \in \mathcal{B}_i$ if $F_i \neq X_i\}$ is a filter base.) The filter it generates is the <u>product filter</u>.

$\pi_i^{-1}(\mathfrak{F}_i) := \{\pi_i^{-1} F_i \mid F_i \in \mathfrak{F}_i\}$ defines a filter base on $\prod X_i$. The product filter is finer than each $\pi_i^{-1}(\mathfrak{F}_i)$. In fact, it is their lub. Moreover the direct image under π_i of the product is $\mathfrak{F}_i$. The product filter is the coarsest whose direct image under π_i is $\mathfrak{F}_i$.

On a product of top spaces, the nbhd filter of a point (x_α) is the product of the nbhd filters of x_α.

<u>No. 8. Elementary filters</u> A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a set X is a map $f: \mathbb{N} \to X$. The direct image of the Fréchet filter on $\mathbb{N}$ under such a map is an <u>elementary filter</u>. Observe that $\{x_m \mid m \geq n\}_{n\in\mathbb{N}}$ is a base of this filter. In other words, ~~the~~ elementary filter corresponding to a sequence $\{x_n\}$ ~~consists~~ consists of subsets to which the sequence eventually belongs.

Let $\mathbb{N} \xrightarrow{\varphi} \mathbb{N}$ be an increasing function and f a sequence in X. Then $f\circ\varphi$ is a <u>subsequence</u> of f. The elementary filter corresponding to $f\circ\varphi$ is finer than that corresponding to f.

<u>Propn:</u> If a filter has a countable base, then it is the intersection of elementary filters containing it.

<u>Proof:</u> Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a base of $\mathfrak{F}$. Put $B'_n := B_1 \cap \dots \cap B_n$. Then $\mathcal{B}' = \{B'_n\}$ is also a base for $\mathfrak{F}$. Consider sequences $\{x_n\}$ s/t $x_n \in B'_n$. Then $\mathfrak{F}$ is the intersection of elementary filters $\mathcal{E}$ corresponding to those sequences. Indeed, B'_n belongs to $\mathcal{E}$ (for any n, for any $\mathcal{E}$), so $\mathfrak{F}$ is coarser than the intersection. On the other hand, if $F \notin \mathfrak{F}$, then $F \not\supset B_n\ \forall n$, and we choose x_n s/t $x_n \in B_n \setminus F$. Then F does not belong to $\mathcal{E}$ corresponding to $\{x_n\}$. $\boxtimes$

Exercise/Example: The finite complement filter ~~on an~~ on an uncountable set does not ~~ha~~ admit a countable base. Nevertheless it is coarser than every elementary filter corresponding to a sequence of distinct elts. ~~of~~ (Thus a coarser filter of a filter ~~base with~~ with countable base may not admit a countable base.)