Let $A \subseteq X$ and $\mathfrak{F}$ a filter on X. The trace of $\mathfrak{F}$ on A is a filter on A iff A meets every elt of $\mathfrak{F}$. In this case, the trace is called the <u>induced filter</u> on A. An ultrafilter induces a filter on A iff it contains A as an element. In this case, the induced filter is an ultrafilter. If A meets every elt of a filter $\mathfrak{F}$ on X, the trace on A of a base $\mathfrak{B}$ of $\mathfrak{F}$ is a base of the induced filter.

<u>Example</u>: The nbhd filter of a point x in a top. space X induces a filter on A iff x belongs to the closure of A.

<u>Construction</u> (Every filter is the induced filter of a nbhd filter). Let $\mathfrak{F}$ be a filter on X. Let $Y = X \amalg \{\omega\}$. Define top on Y by means of nbhds. For x in X, let any set containing x be a nbhd. For ω, the nbhds of the form $F \cup \{\omega\}$ where $F \in \mathfrak{F}$. (Check that axioms $V_I - V_{IV}$ are satisfied.) $\mathfrak{F}$ is the filter on X induced by the nbhd filter on Y at ω. ▨

<u>No 6 Direct image & Inverse image of a base of a filter.</u>

$f\mathfrak{B} = \{fF \mid F \in \mathfrak{B}\}$ is a filter base in Y if $\mathfrak{B}$ is a filter base in X & $f: X \to Y$ a set map: The axioms $(F\mathfrak{B}_I)$ & $(F\mathfrak{B}_{II})$ are easily checked to hold for $f\mathfrak{B}$. (Note that $f\mathfrak{F}$ is only a filter base and not in general a filter ~~even if $\mathfrak{F}$~~ for a filter $\mathfrak{F}$ on X.) If $\mathfrak{B}_1$ is finer than $\mathfrak{B}_2$ (as filter bases in X) then $f\mathfrak{B}_1$ is finer than $f\mathfrak{B}_2$. Thus the filter generated by $f\mathfrak{B}$ depends only on the filter $\mathfrak{F}$ that $\mathfrak{B}$ generates & not on the choice of a base $\mathfrak{B}$ for $\mathfrak{F}$.

<u>Propn:</u> If $\mathfrak{B}$ is a base of an ultrafilter on X, then its image $f\mathfrak{B}$ (which is a filter base in Y) generates an ultrafilter on Y.

<u>Proof:</u> Let $Y' \subseteq Y$. Then $f^{-1}Y'$ & $f^{-1}(Y \setminus Y')$ are complements of each other in X. Thus one of them (but not both) contains an element F of $\mathfrak{B}$. Say $F \subseteq f^{-1}Y'$. Then $fF \subseteq Y'$. QED

<u>Special case:</u> $A \subseteq X$. A filter base $\mathfrak{B}$ on A (in particular a filter on A) is a filter base on X (considered as a collection of subsets of X). If $\mathfrak{B}$ generates an ultrafilter on A, it generates an ultrafilter on X.