 No. 4 Ultrafilters

As seen already, every filter $\mathfrak{F}$ is contained in an ultrafilter $\mathfrak{U}$ (for the filters finer than $\mathfrak{F}$ form an inductive set).

__Lemma:__ Suppose $A \subseteq X$ & $\mathfrak{F}$ a filter on X s/t $A \notin \mathfrak{F}$.
Then $\exists$ filter $\mathfrak{F}'$ finer than $\mathfrak{F}$ and s/t $\complement A \in \mathfrak{F}'$

__Proof:__ $\mathfrak{F}' := \{M \subseteq X \mid M \cup A \in \mathfrak{F}\}$ is a filter with the desired properties. $\boxed{}$

__CoR:__ Let $\mathfrak{F}$ be a filter. Then $\mathfrak{F}$ is an ultrafilter $\Leftrightarrow$ $\forall A \subseteq X$ either A or $\complement A$ belongs to $\mathfrak{F}$. __Proof:__ ($\Rightarrow$) if $A \notin \mathfrak{F}$, then construct $\mathfrak{F}'$ as in lemma. ~~Be~~ Since $\mathfrak{F}$ is ultra, we have $\mathfrak{F} = \mathfrak{F}'$. But $\complement A \in \mathfrak{F}'$. ($\Leftarrow$) Suppose not. Let $\mathfrak{F} \subsetneq \mathfrak{F}''$. Choose $A \in \mathfrak{F}'' \setminus \mathfrak{F}$. Construct $\mathfrak{F}'$ finer than $\mathfrak{F}$ s/t $\complement A \in \mathfrak{F}'$.
Now $A \notin \mathfrak{F}$ since $A \notin \mathfrak{F}'$ & $\complement A \notin \mathfrak{F}$ since $\complement A \notin \mathfrak{F}''$. $\boxed{}$

__CoR:__ Suppose $A \cup B \in \mathfrak{U}$ ultrafilter. Then $A \in \mathfrak{U}$ or $B \in \mathfrak{U}$.
__Proof:__ If $A \notin \mathfrak{U}$ & $B \notin \mathfrak{U}$, then $\complement A, \complement B \in \mathfrak{U}$. But $\complement A \cap \complement B = \complement(A \cup B) \in \mathfrak{U}$ ✱.

__CoR:__ Every filter is the intersection of the ultrafilters containing it.
__Proof:__ Suppose $A \notin \mathfrak{F}$ filter. Then $\exists$ filter $\mathfrak{F}'$ finer than $\mathfrak{F}$ s/t $\complement A \in \mathfrak{F}'$.
Any ultrafilter containing $\mathfrak{F}'$ does not contain A. QED

__Example__: (__Trivial Ultrafilter__) The subsets containing a fixed x_0 in X form an ultrafilter (by the criterion in the first corollary above).
These are called __trivial ultrafilters__.

__Caution:__ Trivial ultrafilters are the only ones we explicitly construct.
As to others, we only deal with them indirectly /existentially.