Chapter I. § 6 Filters No.1. Definition of a filter. 

Defn: A filter $\mathcal{F}$ on a set $X$ is a collection of subsets of $X$ subject to the following:
($\mathcal{F}_1$) closed under superset: if $A \in \mathcal{F}$ & $A \subseteq B$, then $B \in \mathcal{F}$ ($\mathcal{F}_2$) $\mathcal{F}$ is closed under finite intersections
($\mathcal{F}_3$) $\phi \notin \mathcal{F}$

Remarks:
- $X \in \mathcal{F}$ (by ($\mathcal{F}_2$) since $X$ is the empty intersection)
- Thus there is no filter on an empty set.
- ($\mathcal{F}_2$) is equivalent to: ($\mathcal{F}_{2a}$) $A \in \mathcal{F}$ & $B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ & ($\mathcal{F}_{2b}$): $X \in \mathcal{F}$

Examples: ① All subsets containing a non-empty subset $A$.
② All nbhds of a point $x$ in $X$ for a topological space $X$.
③ $X$ infinite, $\mathcal{F}$ = subsets with finite complement. When $X = \mathbb{N} = \{0,1,2,\ldots\}$ this filter is called the Fréchet filter.

No.2 Comparison of filters: coarser, finer, comparable.

~~Theorem~~ Any non-empty intersection of filters is also a filter (GLB).
The filter $\{X\}$ is the coarsest of all filters on $X$.

Question: Given a subset $\mathcal{G}$ of $\mathcal{P}(X)$, when is there a filter $\mathcal{F}$ containing $\mathcal{G}$?
Answer: Iff any finite intersection of elements of $\mathcal{G}$ is non-empty.
Proof: $\Rightarrow$ $\because$ of axiom ($\mathcal{F}_2$). $\Leftarrow$ Set $\mathcal{F}$ to be the set of subsets containing a finite intersections of elts of $\mathcal{G}$. Then $\mathcal{F}$ is a filter. ▨

$\mathcal{F}$ as in the proof above is denoted $\mathcal{G}''$ and is called the filter generated by $\mathcal{G}$.
$\mathcal{G}'$ = finite intersections of elts of $\mathcal{G}$. $\mathcal{G}$ is called a subbase of $\mathcal{F} = \mathcal{G}''$.
Clearly $\mathcal{G}''$ is the ~~smallest~~ coarsest filter containing $\mathcal{G}$.

COR $\mathcal{F}$ filter & $A$ subset $\neq \emptyset$. There exists a filter $\mathcal{F}'$ finer than $\mathcal{F}$ and with $A \in \mathcal{F}'$ iff $A$ meets every elt of $\mathcal{F}$.

COR A set $\Phi$ of filters on a non-empty set $X$ has a LUB filter iff $\forall$ finite set $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of elts of $\Phi$ and $\forall$ choice $A_1 \in \mathcal{F}_1, \ldots, A_n \in \mathcal{F}_n$ we have $A_1 \cap \ldots \cap A_n \neq \emptyset$.

COR: The ordered set of filters is inductive. Thus by Zorn's lemma, we have:

Theorem: For any filter $\mathcal{F}$ on a set $X$, there exists a maximal filter finer than $X$. (Maximal filters are called ultrafilters.)