No. 3 Properties peculiar to open mappings.

Propn: TFAE for a set map $f: X \to Y$ between top spaces X & Y:
① f is open ② The image under f of every basic open set is open in Y (for some base)
③ $\forall x \in X$ & $\forall$ nbhd V of x, fV is a nbhd of fx in Y.

Propn: TFAE for an equivalence relation R on a top space X:
① R is open (i.e., $X \xrightarrow{\pi} X/R$ is open) ② $S \subseteq X$ saturated (w.r.t. R) $\Rightarrow$ $\mathring{S}$ saturated
③ $S \subseteq X$ saturated $\Rightarrow$ $\overline{S}$ saturated.

Proof: ② $\Leftrightarrow$ ③: S saturated $\Leftrightarrow$ its complement is saturated. $\mathring{S} = \complement(\overline{\complement S})$, $\overline{S} = \complement(\mathring{\overline{\complement S}})$. ② $\Rightarrow$ ① $\pi^{-1}(\pi U)$ is the saturation of U for any $U \subseteq X$. If U is open $\pi^{-1}(\pi U) \supseteq \mathring{\overline{\pi^{-1}(\pi U)}} \supseteq U$. By hypothesis $\mathring{\overline{\pi^{-1}(\pi(U))}}$ is saturated. So $\pi^{-1}(\pi U) = \mathring{\overline{\pi^{-1}(\pi U)}}$.
① $\Rightarrow$ ② In general, for S saturated, we have $S = \pi^{-1}(\pi S) \supseteq \pi^{-1}(\pi \mathring{S}) \supseteq \mathring{S}$. Under ①, $\pi^{-1}(\pi \mathring{S})$ is open, so $\mathring{S} \supseteq \pi^{-1}(\pi \mathring{S})$. Thus $\mathring{S} = \pi^{-1}(\pi \mathring{S})$. ▨

Propn: Suppose $X \xrightarrow{\pi} X/R$ is open. Then $\forall A^{\text{saturated}}$, we have ① $\pi(\mathring{A}) = \mathring{\overline{\pi A}}$ & ② $\overline{\pi A} = \pi(\overline{A})$

Proof: ① Since A is saturated (and $\mathring{\overline{\pi A}} \subseteq \pi A$), we have $\pi^{-1}(\mathring{\overline{\pi A}}) \subseteq A$. Since π is continuous, $\pi^{-1}(\mathring{\overline{\pi A}}) \subseteq \mathring{A}$. Applying π we get $\mathring{\overline{\pi A}} \subseteq \pi(\mathring{A})$. ~~Now suppose that π is open~~ (Thus this holds in general for $A^{\text{saturated}}$.) Now suppose that π is open. Then $\pi(\mathring{A})$ is open and so $\pi(\mathring{A}) \subseteq \mathring{\overline{\pi A}}$.
② If A is saturated, then $\complement A$ is also saturated (clearly) and $\mathring{A}$ is saturated (by previous propn). Thus (if A is saturated)

$$\overline{\pi A} \underset{\uparrow\ \text{relation between closure \& interior}}{=} \complement(\mathring{\overline{\complement \pi A}}) \underset{\because A \text{ is saturated}}{=} \complement(\mathring{\overline{\pi \complement A}}) \underset{\because \text{ of ①}}{=} \complement(\pi(\mathring{\overline{\complement A}})) \underset{\because \mathring{\overline{\complement A}} \text{ is saturated}}{=} \pi(\complement(\mathring{\overline{\complement A}})) = \pi \overline{A}.$$

QED.

Propn (H) $f_i: X_i \to Y_i$ open, f_i surjective for almost all i (C) $\prod f_i: \prod X_i \to \prod Y_i$ is open.

CoR: (H) Suppose that $X_i \to X_i/R_i$ is open ($\forall i \in I$). Define $\prod R_i$ on $\prod X_i$ by: $(x_i) \sim (x_i') \Leftrightarrow x_i \sim x_i' \ \forall i$
(C) ~~$\prod X_i/\prod R_i$~~ $\prod X_i \to \prod X_i/\prod R_i$ is open & $\prod X_i/\prod R_i \xrightarrow[\text{homeo}]{\sim} \prod(X_i/R_i)$.

Proof: $\prod X_i \to \prod\left(\frac{X_i}{R_i}\right)$ naturally decomposes as $\prod X_i \to \prod X_i/\prod R_i \xrightarrow[\text{bijection}]{} \prod(X_i/R_i)$.
Now $\prod X_i \to \prod(X_i/R_i)$ is open by propn just above. Now apply I§5 No2 Propn. ▨

Example (Exercise) Let R & S be equivalence relations on top spaces X & Y resp. If R & S are open, then $X \times Y/R \times S \cong X/R \times Y/S$ (by Cor). However if R & S are not ~~open~~ both open, then the ~~map~~ continuous bijection $X \times Y/R \times S \to X/R \times Y/S$ need not be a homeomorphism, even if R happens to be trivial. E.g., let $X = \mathbb{Q}$, $Y = \mathbb{Q}$, R trivial on X, S on Y defined by $y \sim y'$ iff both y & y' are integers. Then
$\mathbb{Q} \times \mathbb{Q}/R \times S \to \mathbb{Q}/R \times \mathbb{Q}/S = \mathbb{Q} \times \mathbb{Q}/S$ is not a homeomorphism.