I. §5. Open mappings & closed mappings No. 2 Open/Closed Equiv. relations

X top space, R equiv. relation on X. <u>Def:</u> R is <u>open/closed</u> if $X \xrightarrow{\pi} X/R$ natural map is open/closed (with X/R given the quotient topology). Equivalently if the saturation of any open/closed set is open/closed.

<u>Example:</u> ① Γ a group of homeo acting on X. ~~[illegible]~~ R: $x \sim x'$ if $\exists\, \gamma \in \Gamma$ s/t $x' = \gamma x$. Equivalence classes are orbits under Γ. We write X/Γ for X/R. This relation is open. Example: $\mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation

② Suppose X is obtained by pasting together top. spaces $\{X_\alpha\}$ along subspaces $X_{\alpha\beta}$ by means of bijections $h_{\alpha\beta}: X_{\alpha\beta} \to X_{\beta\alpha}$ (see I §2. No. 5). If each $X_{\alpha\beta}$ is open in X_α and the $h_{\alpha\beta}$ are all homeomorphisms, then the relation R on $\amalg X_\alpha$ is open. (recall $X = \amalg X_\alpha / R$). Given U open in $\amalg X_\alpha$, its saturation intersected with X_α is given by ~~[illegible]~~ $\bigcup_\beta h_{\beta\alpha}(U \cap X_\beta \cap X_{\beta\alpha})$, and each $h_{\beta\alpha}(U \cap X_\beta \cap X_{\beta\alpha})$ is open in $X_{\alpha\beta}$ and so also in X_α.

③ In example 2, (keeping the notation but not the hypothesis) suppose that $X_{\alpha\beta}$ is closed in X_α and $h_{\alpha\beta}$ are all homeomorphisms. Assume further that for each α only finitely many $X_{\alpha\beta}$ are not empty. Then the equivalence relation is closed.

<u>Propn:</u> Let $X \xrightarrow{\pi} X/R \xrightarrow{h} f(X) \xhookrightarrow{i} Y$ be the canonical decomposition of a continuous mapping $f: X \to Y$. TFAE ① f open ② π, h, i are open ③ R is open, h is a homeo, and $f(X)$ is open in Y.
[May replace "open" by "closed" everywhere]

<u>Propn:</u> Let R be open, let $\pi: X \to X/R$. Suppose that A is a subset of X s/t ① A is ~~closed~~ open ~~&~~ OR ② A is saturated. Then R_A is open & A/R_A onto $\pi(A)$ is a homeomorphism. [May replace "open" by "closed" everywhere.]

<u>Proof:</u> If ①, then $A \hookrightarrow X$ is open and hence so is $A \to X/R$. Now apply propn above. Suppose ② holds. Then ~~$A \hookrightarrow X \to X/R$~~ $A \xrightarrow{\pi} \pi(A)$ is open. [$\because$ if $X \xrightarrow{f} Y$ is open & $T \subseteq Y$, then ~~[illegible]~~ $f^{-1}(T) \xrightarrow{f} T$ is open; apply this with $f = \pi: X \to X/R$ and $T = \pi(A)$: observe $\pi^{-1}(\pi(A)) = A$ because A is saturated.] Now $A \to A/R_A \hookrightarrow \pi(A)$ ~~[illegible]~~ is the canonical decomposition of ~~$A \hookrightarrow X \xrightarrow{\pi} X/R$~~ $A \xrightarrow{\pi} \pi(A)$. Now apply the previous proposition. QED