No. 3 Closures in products.

Propn: $\overline{\prod A_i} = \prod \overline{A_i}$. Proof: $\subseteq$ Since the projections are continuous, $pr_i(\overline{\prod A_i}) \subseteq \overline{pr(\prod A_i)} = \overline{A_i}$. $\supseteq$ Let $(b_i) \in \prod \overline{A_i}$. Let U be an elementary open set around (b_i). $U = \prod U_i$ with U_i open in X_i and $U_i = X_i$ for almost all X_i. Since U_i is open around $b_i \in \overline{A_i}$, we have $U_i \cap A_i \neq \emptyset$. Choose $a_i \in U_i \cap A_i$.

Thus $(a_i) \in \prod A_i \cap U$. QED

CoR: $\prod A_i$ is closed in $\prod X_i$ iff each A_i is closed in X_i.

Remark: In contrast, if I is infinite, just because U_i is open $\forall i$, it ~~is nevertheless~~ is not necessary that $\prod U_i$ be open. For $\prod U_i$ to be open, it is necessary that $U_i = X_i$ for almost all i.

Propn: Let (a_i) be any element of the product $\prod X_i$. Let D be the subset consisting of those elts $d \in \prod X_i$ s/t $pr_i(d) = a_i$ for all but finitely many i (depending upon d). Then D is dense in $\prod X_i$.

Proof: ~~Any open set contains~~ The elementary open sets form a basis of $\prod X_i$. Each such ~~element~~ set is of the form $\prod U_i$ with $U_i = X_i$ except for finitely i, and so meets D. QED

No. 4. Inverse limits • Let I be a poset (reflexive, transitive relation). Let $\{X_i\}_{i\in I}$ be an inverse system of top spaces & cont. maps. Then $\varprojlim X_i$ may be defined a sets. We have natural maps $\varprojlim X_i \to X_i$. We give $\varprojlim X_i$ the initial topology w.r.t. $\varprojlim X_i \to X_i$.

- $\varprojlim X_i \subseteq \prod X_i$ as top. spaces: The topology on $\varprojlim X_i$ coincides with the subspace topology induced from $\prod X_i$. (by transitivity of initial topologies).
- If $Y_i \subseteq X_i$, then $\varprojlim Y_i \subseteq \varprojlim X_i$ as top spaces (again by transitivity)
- More generally, if $X_i' \to X_i$ is an inverse system of continuous maps, we get a natural continuous map $\varprojlim X_i' \to \varprojlim X_i$.
- Suppose I is a direct set & J is a cofinal subset. Then $\varprojlim_I X_i \simeq \varprojlim_J X_i$ naturally homeomorphic. $\{ f_\alpha^{-1}(U_\alpha) \mid \alpha \in J,\ U_\alpha \overset{open}{\subseteq} X_\alpha$ (or more generally U_α ~~belo~~ belongs to a base of X_α)$\}$ is a base of the top. of $\varprojlim X_i$.
- CoR: For $A \subseteq \varprojlim X_i$, let A_i be the image of A in X_i. Then A_i (resp $\overline{A_i}$) form an inverse system. $\overline{A} = \bigcap f_i^{-1}(\overline{A_i}) = \varprojlim \overline{A_i}$. In particular, $A = \bigcap f_i^{-1}(\overline{A_i}) = \varprojlim \overline{A_i}$ if A is closed.