No. 6. Quotient space of a subspace

Let X be a top space & R an equivalence relation on X. Let $A \subseteq X$ be a subset. Consider the canonical decomposition of the continuous map $\pi \circ i : A \subseteq X \to X/R$.

We have:

$$\begin{tikzcd} A \arrow[r, two heads, "\pi_A"] \arrow[dr, "i"'] & A/R_A \arrow[r, two heads, "g"] & f(A/R_A) \arrow[r, hook, "f"] & X/R \\ & X \arrow[urr, "\pi"'] & & \end{tikzcd}$$

where f = the set map $A/R_A \to X/R$ s/t $f \circ \pi_A = \pi \circ i$.

R_A is the equivalence relation on A given by: $a \underset{R_A}{\sim} a'$ if $a \underset{R}{\sim} a'$ in X.

π_A is the quotient map to A/R_A from A. As we have seen, g is a continuous bijection. It need not be a homeomorphism in general. We have:

<u>Propn</u>: TFAE ① g is a homeomorphism ② Every R_A-saturated open set of A is the trace on A of an R-saturated open set of X ③ Same as ② with "closed" replacing "open" $\boxtimes$

We now give two useful sufficient criteria for g to be a homeomorphism.

<u>~~Propn~~ COR</u>: If A is an R-saturated open/closed set of X then g is a homeomorphism.

<u>Proof</u>: In this case any R_A-saturated subset of A is R-saturated; and any open/closed subset of A is open/closed in X. $\boxtimes$

<u>~~Propn~~ COR</u>: If $\exists$ a continuous map $X \xrightarrow{h} A$ s/t $x \sim hx$ $\forall x \in X$, then the canonical map $A/R_A \xrightarrow{j \circ g} X/R$ is a homeomorphism (both j and g are homeomorphisms)

<u>Proof</u>: The map $X \xrightarrow{h} A \to A/R_A$ factors via X/R. This gives a continuous map $X/R \to A/R_A$ which is the inverse of $j \circ g$. $\boxtimes$

<u>Example</u>: Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ($x \sim y$ if $x - y \in \mathbb{Z}$). Let $A = [0,1]$. We claim that ~~we apply the proposition~~. $A/R_A \underset{\text{homeo}}{\cong} \mathbb{R}/\mathbb{Z} = \mathbb{T}$. ~~We apply~~ For the proof, ~~we apply the~~ we first observe that ~~Let F be a closed subset of A. Then F is also~~ $A/R_A \to \mathbb{R}/\mathbb{Z}$ is a bijection ($\because$ A meets every R-equivalence class). Now let F be an R_A-saturated closed subset of A. Then F is closed in $\mathbb{R}$. $F + n$, $n \in \mathbb{Z}$, form a LF family of closed sets. Hence their union is closed. It is clearly R-saturated. Thus $F = A \cap \left(\bigcup_{n \in \mathbb{Z}} F + n\right)$. So condition ③ of the propn is satisfied. $\boxtimes$