No. 4 Quotient Spaces Let R be an equivalence relation on a top space X. A subset S of X is saturated w.r.t. R if it is a union of equivalence classes of R. Recall that the quotient topology on X/R is defined as the final topology w.r.t. $X \xrightarrow{\pi} X/R$. This means: ① π is continuous ② For any continuous map $f: X \to Y$ which factors as $g \circ \pi$ as a set map, the map $g: X/R \to Y$ is continuous.

Properties of the quotient topology: ① $A \subseteq X/R$ is open/closed iff $\pi^{-1}(A)$ is open/closed in X. ② $\exists$ 1-1 correspondence between saturated open/closed sets of X and open/closed sets of X/R: $\left\{\begin{matrix}\text{saturated open/closed}\\ \text{subsets of } X\end{matrix}\right\} \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} \left\{\begin{matrix}\text{open/closed subsets}\\ \text{of } X/R\end{matrix}\right\}$

③ $\exists$ 1-1 correspondence between continuous fns out of X that factor as set maps through π and continuous fns out of X/R:

$$\left\{\begin{matrix}\text{Continuous fns } f: X \to Y \text{ s/t } \exists \text{ a}\\ \text{factorization as set maps } f = f' \circ \pi\end{matrix}\right\} \underset{g\circ\pi \,\leftarrow\!\shortmid\, g}{\overset{f \,\mapsto\, f'}{\rightleftarrows}} \left\{\text{Continuous fns } g: X/R \to Y\right\}$$

Example: On $\mathbb{R}$, define $\sim$ by: $x \sim y$ if $x - y \in \mathbb{Z}$. The quotient space $\mathbb{R}/\sim$ is called the one-dimensional torus and denoted $\mathbb{T}$. There exists a bijective correspondence between continuous fns on $\mathbb{R}$ that are periodic with period 1 and continuous fns on $\mathbb{T}$.

~~Propn~~ Rmk: Let $X \xrightarrow{f} Y$ be a cont. map. Let R, S be equivalence relations on X & Y respectively and suppose that $x \sim x' \Rightarrow fx \sim fx'$. Then the natural set map $X/R \to Y/S$ is continuous.

Transitivity of quotient spaces Let R, S be equivalence relations on a top space X. Suppose that $x \sim_R x' \Rightarrow x \sim_S x'$. Then the natural bijection $\frac{X/R}{S/R} \leftrightarrow X/S$ is a homeomorphism.