No 1 Subspaces. Let $A \subseteq X^{top}$. Recall that the subspace top on A is defined as the inverse image topology under the inclusion of A in X, or, in other words, the coarsest topology such that the inclusion is continuous, or, in other words, open/closed sets of A are precisely traces in A (= intersections with A) of open/closed sets of X.

Example: The subspace topology on $\mathbb{Z}$ as contained in $\mathbb{Q}^{std}$ is discrete.

Remarks: ① $B \subseteq A \subseteq X^{top}$ subspace of a subspace is the subspace (∵ of transitivity of initial topologies)

② A subbase/base of X intersected with A gives a subbase/base of A.

③ Every open/closed set of A is open/closed in X iff A is open/closed in X.

④ The trace in A of the nbhds in X of $a \in A$ / a FSN in X of $a \in A$ is the set of nbhds / a FSN in A of a.

⑤ For $a \in A$, for every nbhd W of a in A to be a nbhd of a in X it is n&s that A be a nbhd of a in X.

⑥ For $B \subseteq A$, $\overline{B}^{\text{in } A} = \overline{B}^{\text{in } X} \cap A$. Thus B is dense in A iff $\overline{B}^{\text{in } X} \supseteq A$ (or, equivalently, $\overline{B}^{\text{in } X} \supseteq \overline{A}^{\text{in } X}$).

⑦ Corollary of 6 (transitivity of density) dense in dense is dense

Propn: $A^{dense} \subseteq X$ & $a \in A$. If V is a nbhd of a in A, then $\overline{V}^{\text{in } X}$ is a nbhd of a in X. Proof: Write $V \supseteq U \cap A \ni a$ for some open U in X. ∵ U is open we have $\overline{V} \supseteq \overline{U \cap A} \supseteq U \cap \overline{A} = U$ (Earlier propn: if O is open then $\overline{O \cap P} \supseteq O \cap \overline{P}$) QED

Propn. Let $\{A_i\}_{i \in I}$ be a collection of subsets of $X^{\text{top space}}$ s/t either ① the interiors of the A_i cover X or ② the $\{A_i\}$ form a LFF of closed ~~sets~~ sets covering X. Then for $B \subseteq X$, B is open/closed in X iff $B \cap A_i$ is open/closed in each A_i.

Proof: Case ① We prove that B is open assuming that $B \cap A_i$ is open $\forall i$. The "closed" half of the statement by taking complements. We have $B = \bigcup (B \cap \mathring{A}_i)$, since the interiors of A_i cover X. Observe that $B \cap \mathring{A}_i$ is open in $\mathring{A}_i$ since $B \cap A_i$ is open in A_i. But then $B \cap \mathring{A}_i$ is open in X, and we are done. Case ② We prove the "closed" half this time. $B = \bigcup_i (B \cap A_i)$. Now $\{B \cap A_i\}$ is a LF family (since $\{A_i\}$ is LF). $B \cap A_i$ is closed in A_i and so closed in X (since A_i is closed in X). Now B being the union of a LF family of closed sets is closed. ☒

Remark: If $\{U_i\}$ is an open cover of X and $\mathcal{B}_i$ is a base for U_i then $\bigcup \mathcal{B}_i$ is a base of X.