No. 4 Final topologies.

Proposition: $X_i^{\text{top space}} \xrightarrow{f_i} X^{\text{set}}$, $i \in I$, be given. Then there exists a unique finest topology on X s/t all the f_i are continuous. More precisely, there exist a unique

FINAL TOPOLOGY topology on X s/t ① the f_i are all continuous & ② for $g: X \to Z^{\text{top space}}$ g is continuous if and only if $g \circ f_i$ continuous $\forall\, i \in I$.

Proof: Uniqueness. If $X, \mathcal{T}_1$ and $X, \mathcal{T}_2$ were two topologies both satisfying ① & ②, then, because $X, \mathcal{T}_1$ satisfies ① & $X, \mathcal{T}_2$ satisfies ②, it follows that the identity map $X, \mathcal{T}_1 \xrightarrow{id} X, \mathcal{T}_2$ is continuous. Switching the roles of $\mathcal{T}_1$ and $\mathcal{T}_2$, the proof of uniqueness is done.

Existence: Define $U \subseteq X$ to be open if $f_i^{-1}(U)$ is open in X_i $\forall\, i \in I$. This defines a topology on X clearly satisfying ①. If $g: X \to Z$ is continuous, then so are all $g \circ f_i$. Suppose that all $g \circ f_i$ are continuous. Then for $V \overset{\text{open}}{\subseteq} Z$, $(g \circ f_i)^{-1}(V) = f_i^{-1}(g^{-1}V)$ is open $\forall\, i$, and so $g^{-1}V$ is open in X. QED

Remark: For $C \subseteq X$, the set C is closed in X iff $f_i^{-1}(C)$ is closed in X_i $\forall\, i$.

Transitivity of the final topology: analogous to the transitivity of the initial topology

Examples: ① $X \to X/R$ where R is an equivalence relation. The final topology on X/R is the quotient topology. Open/closed sets of X/R are canonical images of open/closed sets of X that are saturated w.r.t. R

② GLB topology: $\mathcal{T}_i$, $i \in I$, be topologies on X. Consider $X, \mathcal{T}_i \xrightarrow{id} X$, $i \in I$. The final topology is the finest that is coarser than all $\mathcal{T}_i$. It is just the intersection in $\mathcal{P}(X)$ of all $\mathcal{T}_i$.

③ Direct sum (disjoint union) of top. spaces: $X_i \longrightarrow \coprod X_i$:= disjoint union of the X_i

$A \subseteq X$ is open/closed $\Leftrightarrow$ $A \cap X_i$ is open/closed $\forall\, i$. In particular, each X_i is open & closed

④ Let X_i, $i \in I$, be subsets of X^{set}. Suppose that X_i are top spaces. Assume that

ⓐ $X_i \cap X_j$ is open (resp. closed) in X_i and X_j ($\forall\, i, j$)

ⓑ The subspace topology on $X_i \cap X_j$ as a subspace of X_i is the same as that as a subset of X_j.

Let X be given the final topology s/t all the $X_i \hookrightarrow X$ are continuous. Then $\forall\, i$ ① X_i is open (resp. closed) in X and ② the top. on X_i is the subspace topology.

Proof: ① is clear since $X_i \cap X_j$ is open (resp. closed) in X_j $\forall\, j$ ② Since $X_i \subseteq X$ is continuous, the original top on X_i is finer than the subspace topology. Now suppose A is open in X_i (resp. closed) (in the original topology). $A \cap X_j$ is open (resp. closed) in $X_i \cap X_j$ ~~(by ⓑ)~~. By ⓐ, $A \cap X_j$ is open (resp. closed) in X_j. Thus A is open (resp. closed) ~~in the final topology~~ in X (and so also in X_i in the subspace topology). ☒

Remark: In ④ Outside of $\bigcup_i X_i$, the final topology on X is discrete. If $S \subseteq X$ s/t $S \cap X_i = \emptyset$ $\forall\, i$, then S is open and closed.

We are not assuming that the X_i are mutually disjoint or that they cover X. The question is: what is the relationship of the subspace top on X_i induced from the final top on X to the original top on X_i? Since $X_i \to X$ is cont, the original top is finer than the subspace top. We now give a sufficient criterion for ~~when~~ the two topologies to coincide.