No. 2 Comparison of topologies

Defn: Let $\mathcal{T}_1$ & $\mathcal{T}_2$ be two topologies on a set X. $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ (or $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$) if $(X,\mathcal{T}_1) \xrightarrow{id} (X,\mathcal{T}_2)$ is continuous.

Remarks ① Inclusion in $\mathcal{P}(X)$ orders topologies on X. The topology $\{\phi, X\}$ is the coarsest and the discrete topology the finest. (The above comparison is just this order.)
② The finer the topology the more open sets, the more closed sets, the more nbhds, the smaller the closure of a set, the bigger the interior of a set, the fewer dense sets, the more continuous mapping out, the fewer continuous mappings in.

No. 3. Initial topologies.

X^{set}. $\{X_i \mid i \in I\}$ collection of top spaces. $f_i : X \to X_i$ set maps. There exists a unique coarsest topology on X s/t each f_i is continuous. More precisely

∃! topology s/t ~~with respect to this topology;~~ ① given a set map $g : Z^{top\,space} \to X$, g is continuous at $z \in Z$ if and only if $f_i \circ g$ is continuous at z $\forall\, i \in I$, and ② the f_i are all continuous.

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Proof: The uniqueness ~~is clear~~: if $(X,\mathcal{T}_1)$ and $(X,\mathcal{T}_2)$ are two topologies both satisfying ① & ②, then the identity map $X,\mathcal{T}_1 \xrightarrow{id} X,\mathcal{T}_2$ is continuous. Since the other way around also holds, uniqueness is proved. For the existence, verify that the topology ~~gen~~ with $\{f_i^{-1}(U_i) \mid i \in I,\ U_i \overset{open}{\subseteq} X_i\}$ as a sub~~space~~base satisfies ① and ②. It is clear that ② holds. For ①, first observe that $f_i \circ g$ is continuous at z whenever g is continuous at z. Now suppose that $f_i \circ g$ are continuous at z. Let V be a nbhd of gz. There exists a finite set J of I and open sets $U_j \subseteq X_j$ for $j \in J$ s/t $V \supseteq \bigcap_{j\in J} f_j^{-1}(U_j) \ni gz$. We have $z \in g^{-1}\left(\bigcap_{j\in J} f_j^{-1}(U_j)\right) = \bigcap_{j \in J} (f_j \circ g)^{-1}(U_j) \subseteq g^{-1}V$.
But $\bigcap_{j\in J} (f_j \circ g)^{-1}(U_j)$ is a nbhd of z (by the continuity of $f_i \circ g$ at z). QED

Transitivity of the initial topology Suppose we have $X \xrightarrow{set\ f_i} X_i^{sets}$, $i \in I$, and for each $i \in I$ $X_i^{set} \xrightarrow{f_{ij}} Y_{ij}^{top\,space}$, $j \in J_i$. Give X_i the initial topology so that all f_{ij} are continuous ($j \in J_i$). Then give X the initial topology so that all f_i are continuous ($i \in I$). This is the same topology as the initial topology on X s/t all f_{ij} are continuous ($i \in I$, $j \in J_i$).

Proof: formal consequence of the "universal property" of the initial topology.

Examples: ① $X \overset{inclusion}{\subseteq} Y^{top\,space}$ The initial topology in this case is the subspace topology.
② More generally $X \xrightarrow{set\ f} Y^{top.\,space}$ The initial topology in this case is the inverse image topology. The open sets of X are precisely the inverse images of the open sets of Y. Thus, for x in X inverse image of a FSN of fx is a FSN for x.
③ LUB of a collection of topologies: (coarsest one finer than each belonging to the collection) given $(X,\mathcal{T}_i)$, $i \in I$, consider $X \xrightarrow{id} (X,\mathcal{T}_i)$. The initial topology of this data is the LUB.
④ $\prod X_i \xrightarrow{\pi_i} X_i^{top\,sp}$. The initial topology in this case is the product topology.