No. 1 Continuous functions $f: X \to X'$ mapping betn top spaces X and X' is continuous at $x \in X$ if $\forall$ nbhd V' of $f(x)$ $\exists$ nbhd V of x s/t $f(V) \subseteq V'$ (equivalently $f^{-1}(V') \supseteq V$). [in other words $f^{-1}(V')$ is a nbhd of x]

Remark: This is the exact translation of the intuition "$f(y)$ is as close to $f(x)$ as we please if y is sufficiently close to x".

It is enough that $f^{-1}(V')$ is a nbhd of x $\forall$ V' belonging to a FSN of $f(x)$.

Propn: $X \xrightarrow{f} X' \xrightarrow{g} X''$ If f is cont. at x and g at $f(x)$, then $g \circ f$ is cont at x.

Defn: $f: X \to X'$ is continuous if it is continuous at points of x in X.

Trivial Examples: identity mapping & constant mapping are continuous. Any mapping from a discrete top space and any into a space where only ϕ and the whole set are open are continuous.

Theorem TFAE for $f: X \to X'$. (~~A for satisfying these~~ Equivalent conditions for continuity)
① f is continuous ② $f(\bar{A}) \subseteq \overline{f(A)}$ $\forall A \overset{\text{subset}}{\subseteq} X$ ③ $f^{-1}(V')$ open for V' open
④ $f^{-1}C'$ is closed for C' closed.

Proof: ① ⇒ ② ⇒ ④ ⇒ ③ ⇒ ① ① ⇒ ②: For $b \in \bar{A}$, there are points of $f(A)$ as close as we please to $f(b)$. ② ⇒ ④ $f(\overline{f^{-1}C'}) \subseteq \overline{C'} = C'$ so $\overline{f^{-1}C'} \subseteq f^{-1}C'$.
④ ⇒ ③ by taking complement: $f^{-1}(X' \setminus C') = X \setminus f^{-1}(C')$. ③ ⇒ ① From definitions. ▨

Remark: For continuity, it is enough that preimages of (elements) a base or even a subbase are open.

Example: $\mathbb{Q} \to \mathbb{Q}$ given by $x \mapsto a + x$ or $x \mapsto ax$ (for $a \in \mathbb{Q}$) is continuous.

Caution: Continuous images of opens need not be open; of closed need not be closed.
E.g. $x \mapsto 1/1+x^2$ on $\mathbb{R}$. The image is $[0, 1]$ which is neither open nor closed.

Theorem: Compositions of continuous maps are continuous. Homeomorphisms are precisely bicontinuous bijections.

Caution: Continuous bijections need not be bicontinuous. E.g. $\mathbb{Q}_{\text{std}} \xrightarrow{\text{identity}} \mathbb{Q}_{\text{discrete}}$

Remark: ① For a continuous bijection $f: X \to X'$ to be a homeomorphism it is enough that, for each x in X and each nbhd V of x, $f(V)$ is a nbhd of fx.

② Let X be a top space. Fix $x_0 \in X$. Define new top. space X_0 with underlying set X as follows. For $x_0 \in X_0$ take the nbhds around x_0 to be those around x_0 in X. For $x \neq x_0$ in X_0, take the nbhds to be all subsets containing x. Then the axioms $V_I - V_{IV}$ are satisfied, so X_0 becomes a top. space. The identity mapping $X_0 \to X$ is continuous. For any map $f: X \to Y$ of X into a top space Y, f is continuous at x_0 if and only if the composition $X_0 \xrightarrow{id} X \xrightarrow{f} Y$ is continuous.