Chapter I Topological Structures §1 Open sets, Nbhds, Closed sets

③

No. 6 Interior, Closure, & Frontier of a set; dense sets

<u>Defn:</u> <u>Interior</u> $\mathring{A}$ of a set A defined by $\mathring{A} := \{a \in A \mid A \text{ is a nbhd of } a\}$

Easy facts: $\mathring{A}$ is the largest open set contained in A.

$A = \mathring{A} \iff A$ is open

$A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$

B is a nbhd of $A \iff A \subseteq \mathring{B}$

$\overbrace{A \cap B}^{\circ} = \mathring{A} \cap \mathring{B}$

~~It is not always true that equality holds in~~ $\overbrace{A \cup B}^{\circ} \supseteq \mathring{A} \cup \mathring{B}$ (Equality does not always hold)

<u>Exterior</u> of set A equals (by defn) any of:
- Interior of the complement
- Complement of the closure (see below)
- $\{x \in X \mid x \text{ has a nbhd that does not meet } A\}$

$\bar{A}$ = smallest closed set containing A

<u>Closure</u> $\bar{A}$ of a set A: $\bar{A} := \{x \in X \mid \text{Every nbhd of } x \text{ meets } A\}$

$\bar{A}$ = complement of the exterior; $A = \bar{A} \iff A$ is closed

<u>Rmk:</u> $\complement(\bar{A}) = \overbrace{\complement A}^{\circ}$. So any proposition on interiors leads by duality to a proposition on closures: for instance: $\overbrace{A \cap B}^{\circ} = \mathring{A} \cap \mathring{B}$. Replace A and B by their complements: $\overbrace{\complement A \cap \complement B}^{\circ} = \overbrace{\complement A}^{\circ} \cap \overbrace{\complement B}^{\circ}$ or $\overbrace{\complement(A \cup B)}^{\circ} = \overbrace{\complement A}^{\circ} \cap \overbrace{\complement B}^{\circ}$.
Thus $\complement(\overline{A \cup B}) = \complement(\bar{A}) \cap \complement(\bar{B})$ or $\complement(\overline{A \cup B}) = \complement(\bar{A} \cup \bar{B})$, or $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

<u>Propn:</u> If A is open, then $A \cap \bar{B} \subseteq \overline{A \cap B}$

<u>Defn:</u> $x \in A$ is an isolated point of A if $\exists$ nbhd V of x s/t $V \cap A = \{x\}$.
An isolated point of X is thus a point s/t $\{x\}$ is open.

<u>Perfect set</u>: a closed set with no isolated points

<u>Frontier</u> of A defined by $\bar{A} \cap \overline{\complement A}$. It is clearly closed.
Frontier of A is the same as that of $\complement A$.
The interior, frontier, and exterior of a set form a partition of X.

<u>Defn:</u> A is <u>dense</u> if $\bar{A} = X$, i.e., if every non-empty open set meets A.

<u>Example:</u> $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, X is the only dense set for X discrete

<u>Propn:</u> If $\mathcal{B}$ is a base, then $\exists$ dense set D s/t $\operatorname{Card}(D) \le \operatorname{Card}(\mathcal{B})$

<u>Proof:</u> Choose one point in every basis element. The collection of these is dense. ☒