Chapter I. Topological Structures §1. Open sets, Nbhds, & Closed sets

No. 3. Fundamental systems of nbhds; bases

Defn: A collection N of nbhds of a point x is a fundamental system of nbhds if $\forall$ nbhd of x there exists an elt of N contained in that nbhd.

A fundamental system of nbhds N around x satisfies the following

(FSN_{II}) $\forall$ finitely many elts of N $\exists$ some elt of N contained in their intersection

(FSN_{III}) Every element of N contains x

& (FSN_{IV}) $\forall$ every V of N $\exists$ $W \subseteq V$ s/t $W \in N$ and $\forall w \in W$ $\exists$ a fundamental nbhd of w contained in V

Given FSNs around every point of x, we can recover the set of all nbhds of x ($\forall x$), namely, these are precisely the supersets of fundamental nbhds of x, and so we can recover the topology itself. We may specify the topology by giving sets $\forall x$ that satisfy FSN_{II}, FSN_{III}, FSN_{IV}.

Defn: A collection $\mathcal{B}$ of open sets of a top. space X is a base if every open set is a union of elements of $\mathcal{B}$.

A base satisfies the following: In particular X is ~~the~~ a union of basis elts.

(B_I) The intersection of ~~any two~~ finitely many basis elements is the union of basis elements.

The topology may be recovered from the base: open sets are just unions of basis elts. We may even specify the topology by giving a collection of subsets satisfying (B_I) and saying that the collection is a basis.

Propn (relation between base and fundamental system of nbhds) Elements of a base containing a fixed elt $x \in X$ form a FSN around x. Conversely, if for a collection of open sets, those containing a fixed elt x form an FSN around x $\forall x \in X$, then the collection is a base.

Example: Singleton subsets form a base for the discrete topology.
Open bounded intervals form a base for the std topology on $\mathbb{Q}/\mathbb{R}$.

No. 4 Closed Sets (O_I') (Arbitrary) intersections of closed sets are closed
(O_{II}') Finite unions of closed sets are closed

Example: The union of the closed sets $[\frac{1}{n}, \infty)$ as n varies over $1, 2, \ldots$ is $(0, \infty)$ which is not closed.

No. 5 Locally Finite families. Defn: $\forall x$ in X $\exists$ a nbhd of x ~~open set around x~~ that meets only finitely many members of the family.

Example: $(-\infty, 1), (0, \infty), (1, \infty), (2, \infty), \ldots$ form a LF open cover of $\mathbb{Q}$

Propn: Union of a LF family of closed sets is closed.

Proof: $x \notin \bigcup C_i$. Choose nbhd V of x s/t $V \cap (\bigcup C_i) = (V \cap C_1) \cup \ldots \cup (V \cap C_n)$
Clearly $x \in V \setminus (\bigcup C_i) = V \setminus (C_1 \cup \ldots \cup C_n)$. But $C_1 \cup \ldots \cup C_n$ is closed, so $V \setminus (C_1 \cup \ldots \cup C_n)$ is a nbhd of x. It's a nbhd not meeting $\bigcup C_i$. ☐