I Topological Structures §1 Open sets; Nbhds; Closed sets

No 1 Open sets. Defn: A topology on a set X consists of a collection of subsets of X, which are called open, subject to the following axioms:

(O_I) Closed under arbitrary unions (O_{II}) Closed under finite intersections

Observe: ϕ belongs to the collection ∵ of O_I and X because of O_{II}.

$(O_{II}) \Longleftrightarrow (O_{IIa}) + (O_{IIb})$, where ($O_{IIa}$) the intersection of two closed sets of the collection belongs to the collection. & (O_{IIb}) X belongs to the collection.

Extreme examples: $\{\phi, X\}$ and $\mathcal{P}(X)$ (the latter is called the discrete topology)

Defn: Homeomorphism: a bijection that maps open sets to open sets and under which inverse images of open sets are open.

No 2 Neighborhoods Defn: $A \subseteq X$. A nbhd of A is any subset of X containing an open set containing A.

A nbhd of A is also a neighborhood of any subset of A.

Propn: A subset ~~set~~ is a nbhd of each of its points if and only if it is open.

Remark: Neighborhoods allow us to speak of points "close enough" to a given point and points "as close as we please". For example, the above proposition can be stated as: a subset of X is open if ~~[illegible]~~ and only if for every point in it points that are close enought to that point belong to the set.

The set of neighborhoods of a point x satisfy the following axioms:

(V_I) Closed under passing to a superset (V_{II}) Closed under finite intersection (in particular, X belongs to this collection)

(V_{III}) x belongs to each nbhd &

(V_{IV}) for V a nbhd of x, there exists a nbhd W of x s/t V is a nbhd of w $\forall w \in W$

The topology is determined if we are given the neighborhoods of all points (by Propn above). We may also specify it by means of the nbhds:

Propn: If $\forall x \in X$ there exists subsets of X called "nbhds of x" satisfying (V_I)–(V_{IV}), then there exists a ! topology on X s/t "nbhds of x" are indeed the nbhds of x in the topology.

Example: Topology on $\mathbb{Q}$ and on $\mathbb{R}$. Open sets: Unions of bounded open intervals

Alternatively: Nbhds of a point: Subsets containing ~~a~~ bounded open interval around the point.