Exercise Set 4.

1. What is the gcd of the integers 128 and 231? Find integers a & b s/t $128a + 231b$ equals this gcd.

2. Find a polynomial that leaves remainder j upon division by $t-j$ for $j = 1, 2, 3, 4$. How unique is such a polynomial?

3. Find a polynomial (in the ~~variable~~ variable t) that is $j \bmod (t-j)^j$ for $j = 1, 2, 3, 4$.

4. Describe all matrices that are polynomials in the matrix

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

5. Find the diagonalizable and nilpotent parts of the following matrices (over $\mathbb{C}$): $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

In each case, write these parts as polynomials (without constant term) in the given matrix.

6. Compute the exponential of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

7. Prove that if an $n \times n$ matrix has n distinct eigenvalues then it is diagonalizable.

8. Prove (from first principles) that a real symmetric 2×2 matrix is similar over $\mathbb{R}$ to a diagonal matrix.

9. Let A be a fixed $n \times n$ matrix. Let l_A be the "left multiplication by A" operator on the space of all $n \times n$ matrices. Do A and l_A have the same eigenvalues? What are the characteristic & minimal polynomials of l_A (in terms of data of A). Find the ~~semisimple~~ diagonalizable and nilpotent parts of l_A.

10. Let A be fixed $n \times n$ complex matrix. Consider the operator $\varphi_A : B \mapsto AB - BA$ on $n \times n$ complex matrices. What are the eigenvalues of φ_A? The characteristic polynomial of φ_A? The diagonalizable and nilpotent parts of φ_A? (All in terms of data of A.)