Exercise Set 3

In these, we give a proof of the C-H Theorem and the primary decomposition theorem (see 7 below). This proof of the C-H does not use the determinant trick. The results proved on the way are interesting in their own right. (We assume that the field is alg. closed but the proof can be made to work in general.)

Notation: Let T be a linear endomorphism of a finite dimensional vector space V. Let W be a T-invariant subspace of V. Let $\Phi_T(t) = (t-\lambda_1)^{\ell_1}\cdots(t-\lambda_k)^{\ell_k}$ be the characteristic polynomial of T.

1. Let $f(t)$ be a polynomial such that $f(T)$ is identically zero on V. Then, for any polynomial $g(t)$ coprime to $f(t)$, the operator $g(T)$ operates bijectively ~~hence bijectively~~ on V. [Hint: Write $a(t)f(t)+b(t)g(t)=1$, so $a(T)f(T)+b(T)g(T)=I$. Thus $v = g(T)(b(T)v)$ for any v in V. We also see injectivity easily.] $\boxed{\dim V<\infty \text{ not needed here}}$

2. Any eigenvalue of T on V/W is also an eigenvalue of T on V. [Hint: This follows immediately from the next item but here is a direct proof. Let λ be an eigenvalue of T on V/W. If it is also an eigenvalue of T on W, then of course we are done. If not, then $(T-\lambda)$ acts invertibly on W. Suppose $v\in V/W$ with $v\notin W$ and $(T-\lambda)v\in W$. Then write $(T-\lambda)v = (T-\lambda)\omega$ with $\omega\in W$. Thus $(T-\lambda)(v-\omega)=0$, so we're done.]

3. $\Phi_{T,V}(t) = \Phi_{T,W}(t)\cdot\Phi_{T,V/W}(t)$. In particular, if $V = W\oplus X$ with X also T-invariant, then $\Phi_{T,V}(t) = \Phi_{T,W}(t)\,\Phi_{T,X}(t)$.

4. If $f_1(t),\dots,f_s(t)$ are pairwise coprime polynomials, then the sum $\operatorname{Ker} f_1(T)+\dots+\operatorname{Ker} f_s(T)$ is direct. [Let $v_1+\dots+v_s=0$ with $v_j\in\operatorname{Ker} f_j(T)$. Then $f_1(T)v_2+\dots+f_1(T)v_s=0$. By induction on s, $f_1(T)v_j=0$. By item 1 above, $f_1(T)$ acts bijectively on $\operatorname{Ker} f_j(T)$ for $j\ge 2$, so $v_j=0$ $\forall j$.]

⑤ For a scalar μ, $E_T(\mu) := \bigcup_{n\geq 0} \mathrm{Ker}\,(T-\mu)^n$ is the <u>generalized eigenspace</u>. [If $\mathrm{Ker}\,(T-\mu)^j = \mathrm{Ker}\,(T-\mu)^{j+1}$, then $\mathrm{Ker}\,(T-\mu)^j = \mathrm{Ker}\,(T-\mu)^k$ for all $k \geq j$, so $E_T(\mu) = \mathrm{Ker}\,(T-\mu)^j$ for large j. This also shows that $E_T(\mu) = \mathrm{Ker}\,(T-\mu)^d$ for $d \geq \dim E_T(\mu)$.] Show that

$$E_T(\lambda_1) \oplus \cdots \oplus E_T(\lambda_k) = V.$$

[By item ④, the sum in the LHS is direct. Suppose that $W := LHS \subsetneq V$. By ②, any eigenvalue of T on V/W must be one of $\lambda_1, \dots, \lambda_k$. Let $0 \neq v \in V/W$ with $(T-\lambda_j)v \in W$ for some fixed j. Since $(T-\lambda_j)$ acts invertibly on $E_T(\lambda_i)$, $i \neq j$, [by item ①] we have $(T-\lambda_j)v = w_j + \sum_{i\neq j}(T-\lambda_i)w_i$ with $w_j \in E_T(\lambda_j)$ and $w_i \in E_T(\lambda_i)$. This means $(T-\lambda_j)(v - \sum_{i \neq j} w_i)$ belongs to $E_T(\lambda_j)$, so $v - \sum_{i\neq j} w_i \in E_T(\lambda_j) \Rightarrow v \in W$. $\times$]

⑥ Show that $E_T(\lambda_j) = \mathrm{Ker}\,(T-\lambda_j)^{r_j}$

[Clearly $\supseteq$. By ⑤ and ③, $\cancel{\Phi_T(E_T(\lambda_1)) \cdots \Phi_T(E_T(\lambda_k)) = \Phi_T(t)}$

$\Phi_{T, E_T(\lambda_1)}(t) \cdots \Phi_{T, E_T(\lambda_k)}(t) = \Phi_T(t)$. By item ① of Exercise set ①, $\Phi_{T-\lambda_j, E_T(\lambda_j)}(t) = t^{\dim E_T(\lambda_j)}$, so $\Phi_{T, E_T(\lambda_j)}(t) = (t-\lambda_j)^{\dim E_T(\lambda_j)}$.

Thus $\dim E_T(\lambda_j) = r_j$. See parenthetical remark after definition of generalized eigenspace in item ⑤ above.]

⑦ Putting ⑤ and ⑥ together, we get:

$$\mathrm{Ker}\,(T-\lambda_1)^{r_1} \oplus \cdots \oplus \mathrm{Ker}\,(T-\lambda_k)^{r_k} = V$$

The C-H theorem is now an immediate consequence.

⑧ Let s_j be the least integer s/t $\mathrm{Ker}\,(T-\lambda_j)^{s_j} = E_T(\lambda_j)$.

Then $\mathrm{Ker}\,(T-\lambda_1)^{s_1} \oplus \cdots \oplus \mathrm{Ker}\,(T-\lambda_k)^{s_k} = V$.

The minimal polynomial of T on V is $(t-\lambda_1)^{s_1} \cdots (t-\lambda_k)^{s_k}$.