Exercise Set 2 (1) Computing the exponential of an $n\times n$ matrix.

$$\exp(A) := I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots + \frac{A^j}{j!} + \dots$$

Write $\exp(t) = \chi_A(t)\, g(t) + r(t)$ where $\chi_A(t)$ denotes the char. polynomial of A, $r(t) := r_0 + r_1 t + \dots + r_{n-1} t^{n-1}$ polynomial of degree $n-1$.

The idea is to determine $r(t)$ and thereby compute $\exp(A)$:

$$\exp(A) = r_0 + r_1 A + \dots + r_{n-1} A^{n-1} \quad (\text{since } \chi_A(A) = 0 \text{ by C-H})$$

To compute $r(t)$, plug-in eigenvalues of A. If A has n distinct eigenvalues, say $\lambda_1, \dots, \lambda_n$ then of course $r_0, \dots, r_{n-1}$ are found by solving:

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} \\ e^{\lambda_2} \\ \vdots \\ e^{\lambda_n} \end{bmatrix}$$

If there are repeated eigenvalues, then we differentiate several times as required and plugin the eigenvalues. Illustration:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \chi_A(t) = (t-1)^2(t-2) \quad \text{Write } \exp(t) = (t-1)^2(t-2)\, g(t) + r(t)$$
$$\text{where } r(t) = r_0 + r_1 t + r_2 t^2$$

To determine $r(t)$: Put $t=1$: $e = r_0 + r_1 + r_2$;
Put $t=2$: $e^2 = r_0 + 2r_1 + 4r_2$; differentiate and then put $t=1$:
$e = r_1 + 2r_2$. Thus we have
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} e \\ e^2 \\ e \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & e \\ 1 & 2 & 4 & e^2 \\ 0 & 1 & 2 & e \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & e \\ 0 & 1 & 3 & e^2-e \\ 0 & 1 & 2 & e \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & e \\ 0 & 0 & 1 & e^2-2e \\ 0 & 1 & 2 & e \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & e \\ 0 & 1 & 2 & e \\ 0 & 0 & 1 & e^2-2e \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3e-e^2 \\ 0 & 1 & 0 & 5e-2e^2 \\ 0 & 0 & 1 & e^2-2e \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & e^2-2e \\ 0 & 1 & 0 & 5e-2e^2 \\ 0 & 0 & 1 & e^2-2e \end{array}\right]$$

$$\text{Thus } \exp(A) = (e^2-2e) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (5e-2e^2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} + (e^2-2e) \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{= A^2}$$

$$= \begin{bmatrix} e & e & e^2-2e \\ 0 & e & e^2-e \\ 0 & 0 & e^2 \end{bmatrix} \quad \text{QED.}$$

Exercise Set 2 (Continued) ② Find the characteristic polynomial of the $n \times n$ matrix all of whose entries are λ.

③ Let A be a 3×3 real matrix. True or false?: if A is not similar to an upper triangular real matrix, then A is similar to a diagonal complex matrix.

④ True or false: An upper triangular matrix that is similar to a diagonal matrix is itself diagonal.

⑤ Let $\lambda_1, \ldots, \lambda_n$ (possibly repeated) be the eigenvalues of an $n \times n$ matrix A. Let $f(t)$ be a polynomial with complex coefficients. What is the characteristic polynomial of $f(A)$?

⑥ Let V be the space of all $n \times n$ complex matrices. Let A be a fixed $n \times n$ matrix. Let U and T be the linear operators on V given by $T: X \mapsto AX$, $U: X \mapsto AX - XA$. ~~Show~~ Observe that if A is diagonalizable then so are U and T. Determine their eigenvalues (in terms of those of A). Prove that as A varies over diagonal matrices, then the operators $U(A)$ ~~& T(A)~~ are simultaneously diagonalizable [$\mathrm{lll}^{\mathrm{ly}}$ for $T(A)$].

⑦ Let V be ~~the~~ a subspace of 3×3 matrices all of whose elements commute with one another. What is the maximum possible dimension of V.

⑧ If A & B are real $n \times n$ matrices that are similar as complex matrices, then are they similar also as real matrices?

⑨ Let T be an operator on an $n \times n$ dimensional space with n distinct eigenvalues. Then any operator that commutes with T is a polynomial in T.