Exercise Set 1 Cayley-Hamilton theorem

Recall of C-H Theorem: For any $n\times n$ matrix A, $\boxed{\Phi_A(A)=0}$, where $\Phi_A(t) := \det(tI_{n\times n} - A)$ is the characteristic polynomial of A.

1. Prove directly, from first principles, the C-H Theorem for a nilpotent $n\times n$ matrix A. (A nilpotent if $A^k=0$ for some positive integer k.) [Hint: Let T be the linear operator corresponding to A corresponding to a choice of basis of an n-dimensional vector space V. Consider $V \supseteq TV \supseteq T^2V \supseteq T^3V \supseteq \ldots$ If $T^kV = T^{k+1}V$, then T^kV is the "stable value", ~~thus T^k~~ which is 0 because T is nilpotent. Thus $T^kV \supsetneq T^{k+1}V$ whenever $T^kV \neq 0$. But since the dimension of V is n, we see easily that $T^nV=0$. On the other hand, since zero is the only eigenvalue of A, it is clear that $\Phi_A(t) = t^n$.]

2. Let T be a nilpotent linear operator on a finite dimensional vector space V. Show that $\dim T^kV - \dim T^{k+1}V \geq \dim T^{k+1}V - \dim T^{k+2}V$ for any $k \geq 0$ ($T^0V := V$).

3. Show that $\Phi_{AB}(t) = \Phi_{BA}(t)$ for $n\times n$ matrices A & B. [Hint: Think of A as fixed and B as varying. Note that $tI - AB$ and $tI - BA$ are similar when B is invertible, and so ~~equality~~ $\Phi_{AB}(t) = \Phi_{BA}(t)$ in this case. But $B \mapsto \Phi_{AB}(t)$ & $B \mapsto \Phi_{BA}(t)$ are continuous fns and invertible matrices are dense in the space of all (say complex) $n\times n$ matrices.]

4. (Generalization of 3) If A is an $m\times n$ and B is an $n\times m$ matrix then $t^n\Phi_{AB}(t) = t^m\Phi_{BA}(t)$. [Proof:

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}.]$$

5. The association $A \mapsto$ minimal polynomial of A is ~~not~~ NOT continuous on $n\times n$ matrices.

6. Let A be an $n\times n$ matrix and $\operatorname{adj} A$ its adjoint. What are the possibilities for the rank of $\operatorname{adj} A$?

7. Show that every root of the minimal polynomial of an $n\times n$ matrix A is a root of $\Phi_A(t)$ and vice-versa. (In fact, every irreducible factor of the minimal polynomial (in case the field is not alg closed) is an irreducible factor of $\Phi_A(t)$ and vice-versa.) These factors need not be linear. Note that the C-H theorem can be restated as: the minimal polynomial divides $\Phi_A(t)$.

8. Any real $n\times n$ matrix A can also be considered as a complex matrix. Note that $\Phi_A(t)$ does not depend on the point of view. How about the minimal polynomial of A? Does it depend upon the point of view?

9. Suppose that a real $n\times n$ matrix A has characteristic polynomial $(t-3)^4(t-4)^3(t-5)^5(t^2+1)^2$. What is n? How many different possibilities are there for the minimal polynomial of A? List them.

10. Compute $\Phi_A(t)$ for the $n\times n$ matrix A all of whose entries are 2.

11. ~~Among~~ Show that among all 2×2 matrices over complex numbers those having two distinct roots for the characteristic polynomial ~~are~~ form a dense open set. Do they form a connected set? How about for $n\times n$ complex matrices?