Schur Functions S_λ

Summary: $\boxed{S_\lambda := a_{\lambda+\delta}/a_\delta}$ Weyl Character Formula

$\boxed{S_\lambda,\ \lambda\vdash \text{ form a } \mathbb{Z}\text{-basis for } \Lambda}$. In fact, $\boxed{S_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu}$ — (1)

$\boxed{K_{\lambda\mu} = \text{"Kostka numbers"} = \#\text{ of standard tableaux of shape } \lambda \text{ and content } \mu}$ (Therefore non-negative).

$\boxed{\langle S_\lambda, S_\mu\rangle = \delta_{\lambda\mu}}$ (2) Orthonormality. Equivalently $\boxed{\sum_{\lambda\vdash} S_\lambda(x)S_\lambda(y) = \prod(x,y)}$ (2)

S_λ are determined (over determined) from (1) & (2). In fact, S_λ are uniquely determined by (1) and (2') where (2'): $\langle S_\lambda, S_\mu\rangle = 0$ for $\mu<\lambda$.

Proof: $S_\lambda - m_\lambda = \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu \in \langle S_\mu \mid \mu<\lambda\rangle := \mathbb{Z}$-span of S_μ by (1)

Now, (2') means $S_\lambda - m_\lambda$ is uniquely determined: ~~that is~~,

$S_\lambda - m_\lambda = -\sum_{\mu<\lambda} (\langle m_\lambda, S_\mu\rangle / \langle S_\mu, S_\mu\rangle) S_\mu$.

$\boxed{\langle S_\lambda, p_\mu\rangle = \text{value of the irr. char } \chi_\lambda \text{ of } \mathfrak{S}_n \text{ on the conjugacy class of cycle type } \mu,\ \lambda\vdash n,\ \mu\vdash n}$ FROBENIUS CHARACTER FORMULA

$\boxed{S_\lambda = \det(h_{\lambda_i+j-i})_{1\le i,j\le n}}$ $n \ge \#$ of parts in λ JACOBI-TRUDI

$h_r := 0$ if $r<0$, $h_0 := 1$

$\boxed{S_\lambda = \det(e_{\lambda'_i-i+j})_{1\le i,j\le m}}$ $m \ge \#$ of parts in λ' GIAMBELLI

$e_r := 0$ if $r<0$, $e_0 := 1$

$\boxed{\omega(S_\lambda) = S_{\lambda'}}$ This follows from Jacobi-Trudi & Giambelli, for $\omega h_r = e_r$