(NEWTON'S) Power sum Symmetric functions

$p_d := \sum x_i^d = x_1^d + x_2^d + \dots \qquad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots = p_1^{m_1} p_2^{m_2} \cdots$

$$\boxed{P(t) = \sum_{i \geq 1} \frac{x_i}{(1-x_i t)}}$$

$$\overset{\circ}{\text{ii}} \quad \sum p_{d+1} t^d$$

Let us relate $P(t)$ to $E(t) = \prod_{i\geq 1}(1+x_i t)$ & $H(t) = \dfrac{1}{\prod_{i \geq 1}(1-x_i t)}$

We have $E'(t) = \left(\sum_{i\geq 1} \dfrac{x_i}{1+x_i t}\right) E(t)$ & $H'(t) = \left(\sum_i \dfrac{x_i}{(1-x_i t)}\right) H(t)$

i.e., $\boxed{E'(t) = P(-t)E(t)}$ & $\boxed{H'(t) = P(t)H(t)}$

Comparing coefficients of t^{d-1} in the latter, we get:

$$\boxed{d h_d = p_1 h_{d-1} + p_2 h_{d-2} + \dots + p_{d-1} h_1 + p_d}$$

Thus $h_d \in \mathbb{Q}[p_1, \dots, p_d]$ and $p_d \in \mathbb{Z}[h_1, \dots, h_d]$

p_λ, $\lambda \vdash$, form $\mathbb{Q}$-basis for $\Lambda \otimes \mathbb{Q}$.
$\Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$ and $p_1, p_2, \dots$ are ~~linea~~ alg. independent

Since $h_d \in \mathbb{Q}[p_1, \dots, p_d]$, it is clear that $\Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$
Also $p_1, \dots, p_d$ cannot be algebraically dependent (for any d) because $\mathbb{Q}[p_1, \dots, p_d] = \mathbb{Q}[h_1, \dots, h_d]$, and $h_1, \dots, h_d$ are algebraically independent.

Expressing h_n as a polynomial in the p_j:

$H'(t) = P(t)H(t) \Rightarrow H(t)/H'(t) = P(t) \Rightarrow d/dt \log H(t) = P(t)$

Integrating which, we get $\sum_{d \geq 1} p_d t^d/d = \log H(t)$.

Exponentiating, we get $H(t) = \exp\left(\sum_{d\geq 1} \dfrac{p_d t^d}{d}\right) = \prod_{d \geq 1} \exp\left(\dfrac{p_d t^d}{d}\right)$

Thus $H(t) = \sum_{\lambda} \left(\dfrac{p_1^{m_1}}{1^{m_1} m_1!} \cdot \dfrac{p_2^{m_2}}{2^{m_2} m_2!} \cdots\right) t^{|\lambda|}$ where $\lambda = 1^{m_1} 2^{m_2} \dots$

Set $z_\lambda := 1^{m_1} m_1! \cdot 2^{m_2} m_2! \cdots$ = # of the centralizer in $\mathfrak{S}_{|\lambda|}$ of a permutation of cycle type λ.

Thus $\boxed{h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda}$

Behaviour w.r.t. the involution ω:

Applying ω to $H'(t) = P(t) H(t)$ we get $E'(t) = \omega P(t) E(t)$.
Comparing with $E'(t) = P(-t)E(t)$, we get $\omega P(t) = P(-t)$

or $\boxed{\omega p_d = (-1)^{d-1} p_d}$