# Symmetric Functions

Notes of two talks given in RT Seminar, IMSc, August 2012

Reference: "Macdonald: Symmetric fns & orthogonal polynomials" Macdonald. AMS. University Lecture Series, Vol. 12. Chapter 1.

$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \oplus \Lambda_n^d$

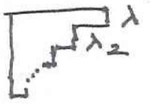
$\Lambda^d := \varprojlim \Lambda_n^d$ ; $\Lambda = \oplus \Lambda^d$

Partition $\lambda \vdash$ $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots$ eventually 0

$|\lambda| := \lambda_1 + \lambda_2 + \ldots$

Also $1^{m_1} 2^{m_2} \ldots$

$\lambda \geq \mu \overset{\text{def}}{\Longleftrightarrow} \lambda_1 \geq \mu_1$ & $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ & $\lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3$

Also 


$\lambda'$ := conjugate of $\lambda$: $\lambda'_i :=$ # of $\lambda_j$ s/t $\lambda_j \geq i$

## MONOMIAL SYMMETRIC FUNCTIONS: THE $m_\lambda$

$x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \ldots$ $\quad m_\lambda := \overleftrightarrow{x^\lambda} :=$ sum of all monomials in the $\mathfrak{S}_\infty$-orbit of $x^\lambda$

Note: Given a monomial of degree $d$ (in $\mathbb{Z}[x_1, x_2, \ldots]$) it is $\mathfrak{S}_\infty$-conjugate to a unique monomial $x^\lambda$ with $\lambda \vdash d$. Given a monomial of degree $d$ in $\mathbb{Z}[x_1, \ldots, x_n]$, it is $\mathfrak{S}_n$-conjugate to a unique monomial $x^\lambda$ with $\lambda \vdash d$ and # of parts of $\lambda \leq n$.

($\lambda$ is the arrangement in non-increasing order of the expns in the monomial)

COR: $\Lambda^d$ has $\mathbb{Z}$-basis $m_\lambda$ with $\lambda \vdash d$. $\boxed{\Lambda \text{ has } \mathbb{Z}\text{-basis } m^\lambda, \lambda \vdash}$

$\Lambda_n$ has $\mathbb{Z}$-basis $m_\lambda$ with # of parts of $\lambda \leq n$

Note: $\Lambda^d \longrightarrow \Lambda_d^d$ natural map is an isomorphism

## ELEMENTARY SYMMETRIC FUNCTIONS: THE $e_\lambda$, $E(t) := \sum_{d \geq 0} e_d t^d$

$$\boxed{E(t) = \prod_{i \geq 1} (1 + x_i t)} \qquad e_d := m_{(1^d)} = \overleftrightarrow{x_1 x_2 \cdots x_d}$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots = e_1^{m_1} e_2^{m_2} \cdots$$

$\boxed{e_\lambda, \lambda \vdash \text{ form a basis of } \Lambda. \text{ In other words } \Lambda = \mathbb{Z}[e_1, e_2, \ldots] \text{ with } e_1, e_2, \ldots \text{ algebraically independent.}}$

Proof: Imagine expanding out $e_{\lambda'}$. For example, let $\lambda = \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$

Then $\lambda' = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\quad e_{\lambda'} = e_2 e_1^2 = \underset{i<j}{\overleftrightarrow{x_i x_j}} \; \overleftrightarrow{x_k} \; \overleftrightarrow{x_l}$ Identify the term $x_i x_j x_k x_l$ in the expansion with the filling $\begin{array}{|c|c|c|}\hline i & k & l \\ \hline j & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ of the boxes in $\lambda$.

The fillings are s/t: 1 can occur only in 1st row. 2 can occur only in the first 2 rows, 3 in the first 3 rows, etc

Thus $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ arises just once: namely as $x_1 x_2 \cdot x_1 \cdot x_1$.

and any $m_\mu$ occurring in the expansion must be $\leq \lambda$

Thus $\boxed{e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu \text{ with } a_{\lambda\mu} \text{ non-negative}}$