Character theory

The "representation theory" for a group G consists of understanding of the following questions:

- Describe the irreducible characters and the irreps.
- Given a representation, find techniques for decomposing into irreps.
- Plethysm: Decompose into irreducible components the representations associated to a given representation U such as $U \otimes U$, $\mathrm{Sym}^k(U)$, $\wedge^k(\wedge^l(U))$. The problem of decomposing $V \otimes W$ for irreps V and W is called the Clebsch-Gordon problem.

Orthonormality of Irreducible Characters

On complex-valued functions on the group introduce the following hermitian inner product:

$$\langle e, f \rangle := \int_G \overline{e(g)} f(g) \, dg$$

The theorem is the following:

> The characters of the irreducible representations form an orthonormal basis for the space of class functions with this inner product.

Let us first show the orthonormality of the irreducible characters. This part of the proof, as also the other half, is a combination of the averaging technique and Schur's lemma. Given a G-module U, the expression $\mathrm{Av} := \int_G g \, dg$ has meaning as an element of $\mathrm{End}(U)$. Being a projection onto U^G, its trace equals $\dim U^G$. Now let V and W be irreducible representations, and set $U = \mathrm{Hom}(V, W)$. By Schur, U^G has dimension 0 or 1 according as V and W are equivalent or not, so that accordingly Av has trace 0 or 1. On the other hand, $\mathrm{Hom}(V, W)$ being isomorphic to $V^* \otimes W$, the trace of Av is just $\langle \chi(V), \chi(W) \rangle$.

Here are some corollaries:

- The character determines the representation. In fact, the number of copies of an irrep V in a given representation U is just the inner product of the characters of V and U.

- A representation is irreducible if and only if its character has norm 1.

- Each irrep V occurs in the regular representation $\dim V$ times. In particular, we have the following formula:

$$|G| = \text{the sum of the squares of the dimensions of all irreps}$$

- We will show shortly that the number of irreps equals the number of conjugacy classes. Assume this for the moment and consider the square matrix of the character table where each entry is multiplied by $\sqrt{c/|G|}$ where c is the cardinality of the corresponding conjugacy class. The orthonormality of characters amounts to the rows of this matrix being orthonormal. The matrix is thus orthogonal, and the columns are also orthonormal. This gives the following:

$$\sum_{\chi} \overline{\chi(g)}\chi(h) = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not conjugate} \\ |G|/c & \text{if } g \text{ and } h \text{ are conjugate} \end{cases}$$

Let us now show that there are no more linearly independent class functions than irreps. Let ϕ be a class function such that $\langle \phi, \chi \rangle = 0$ for all characters χ. The element $\Phi := \sum \overline{\phi(g)} g$ is, on the one hand, by Schur, acting on any irreducible V as a scalar, but, on the other, has trace 0 as an elment of $\mathrm{End}(V)$. It is therefore zero on V and so also on any module U. (Moral: A scalar matrix in characteristic zero vanishes if its trace does.) But then the action of Φ on 1 in the regular representation is Φ, so that Φ is zero. This shows that ϕ is zero and the theorem is completely proved.

0.1 A projection formula

Let us write down a formula for the G-projection from a G-module U to its isotypical component corresponding to an irrep V. (By Schur, there is only one projection.) But what is an "isotypical component"? For a G-module U and an irrep V, define the isotypical component of U corresponding to V to be the sum of all simple submodules of U isomorphic to V. That U is a sum of its isotypical components follows from complete reducibility and that this sum is direct from complete reducibility and Schur.

Consider $\int_G \overline{\chi_V(g)} g \, dg$ in $\mathrm{End}(U)$. This is a G-map and thus by Schur restricts to a scalar on each irreducible sub W. The trace of this restriction is either 1 or 0 according as W is equivalent or not to V. Thus $\dim V \cdot \int_G \overline{\chi(g)} g \, dg$ is the required formula.

The Example of $\mathfrak{S}_3$

The representation theory of the symmetric group $G = \mathfrak{S}_3$ can be understood by elementary means as follows. Let $\sigma = (12)$ and $\tau = (123)$. The elements $\alpha = (\omega, 1, \omega^2)$ and $\beta = (1, \omega, \omega^2)$ form a basis for the standard representation. We have

$$\tau\alpha = \omega\alpha, \quad \tau\beta = \omega^2\beta, \quad \sigma\alpha = \beta, \quad \sigma\beta = \alpha.$$

Let U be any G-module. Let $U = U_1 \oplus U_\omega \oplus U_{\omega^2}$ be the decomposition of U into eigenspaces for τ. The relation $\tau\sigma = \sigma\tau^2$ shows that σ stabilises U_1 and maps U_ω isomorphically onto U_{ω^2}. Let $U_1 = U_{1,1} \oplus U_{1,-1}$ be the decomposition of U_1 into eigenspaces for σ. Let $v_1, \ldots, v_k$ be a basis of U_ω and set $V_1 := \mathbb{C}v_1 \oplus \mathbb{C}\sigma v_1, \ldots, V_k := \mathbb{C}v_k \oplus \mathbb{C}\sigma v_k$. Then $U_\omega \oplus U_{\omega^2} = V_1 \oplus \cdots \oplus V_k$ and each V_j is isomorphic to the standard representation. The elements $(v_j \pm \sigma v_j)/2$ are eigenvectors for σ with eigenvalues ± 1 and form a basis for V_j. Thus we have

$$U = \mathrm{Trivial}^{\oplus a} \oplus \mathrm{Sign}^{\oplus b} \oplus \mathrm{Standard}^{\oplus c}$$

where

$$\begin{aligned}
a + b &= \text{dimension of the eigenspace } U_1 \text{ of } \tau \\
a + c &= \text{dimension of the } +1 \text{ eigenspace for } \sigma \\
b + c &= \text{dimension of the } -1 \text{ eigenspace for } \sigma \\
c &= \text{dimension of either eigenspace } U_\omega \text{ or } U_{\omega^2}
\end{aligned}$$

Solution of the Clebsch-Gordon problem

The elements $\alpha\otimes\alpha$, $\alpha\otimes\beta$, $\beta\otimes\alpha$, $\beta\otimes\beta$ are eigenvectors for τ with eigenvalues ω^2, 1, 1, ω; the elements $(\alpha\otimes\beta \pm \beta\otimes\alpha)/2$, $(\alpha\otimes\alpha \pm \beta\otimes\beta)/2$ are eigenvectors for σ with eigenvalues ± 1, ± 1. Thus

$$\mathrm{Standard} \otimes \mathrm{Standard} = \mathrm{Trivial} \oplus \mathrm{Sign} \oplus \mathrm{Standard}.$$