SEPARATION AXIOMS AND METRIZABILITY

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By a *neighbourhood* of a subset A (in particular of a singleton and therefore of a point) in a topological space X, we mean a subset of X that contains an open set containing A.[1]

The Hausdorff property.

(1) The following conditions on a topological space X are equivalent. We say that X is *Hausdorff* if they hold.
  (H0) Any two distinct points have disjoint neighbourhoods.
  (H1) The intersection of closed neighbourhoods of any point consists only of that point.
  (H3) The diagonal in $X \times X$ is closed.
  (H4) The diagonal in X^I for any index set I is closed.

(2) Any subspace of a Hausdorff space is Hausdorff. Any topology that is finer than a Hausdorff topology is Hausdorff.

(3) Let $f : X \to Y$ and $g : X \to Y$ be continuous functions of topological spaces. If Y is Hausdorff, then $\{x \in X \mid f(x) = g(x)\}$ is closed in X. (Hint: The diagonal in $Y \times Y$ is closed and $\{x \in X \mid f(x) = g(x)\}$ is the inverse image of the diagonal under the continuous map $f \times g : X \to Y \times Y$.)

(4) (Principle of extension of identities) Let $f : X \to Y$ and $g : X \to Y$ be continuous functions of topological spaces. If Y is Hausdorff and f, g agree on a dense subset of X, then $f = g$. (Hint: Use item (3).)

(5) The *graph* of a map $f : X \to Y$ is defined to be the subset $\{(x, f(x)) \in X \times Y \mid x \in X\}$. Let $f : X \to Y$ be a continuous map from a topological space X to a Hausdorff topological space Y. Then the graph of f is closed. (Hint: The graph is the inverse image of the diagonal under the map $f \times \mathrm{id} : X \times Y \to Y \times Y$.)

(6) If for every pair x and x' of distinct points in X there exists a continuous map $f : X \to Y$ with Y a Hausdorff topological space and $f(x) \neq f(x')$, then X is Hausdorff. In particular, if continuous real valued functions on a space X separate points (that is, given two distinct points there exists a continuous real valued function that takes distinct values on them), then X is Hausdorff.

(7) Singletons are closed in a Hausdorff space. Thus any finite subset of a Hausdorff space is closed, and a Hausdorff topology on a finite set is discrete.

(8) If every point of a topological space X has a closed neighbourhood that is Hausdorff (as a subspace), then X is Hausdorff. Give an example to show that it is possible in a non-Hausdorff space for every point to have an open neighbourhood that is Hausdorff. (Hint: Consider the quotient space $[-1, 1]/\sim$ where $x \sim -x$ for $x \neq \pm 1$.)

(9) The product of Hausdorff spaces is Hausdorff. Conversely, if a product of non-empty spaces is Hausdorff, then so is each factor.

An important non-Hausdorff topology. Let $X := \mathbb{C}^n$, where n is a positive integer. Let $R := \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables with coefficients in $\mathbb{C}$. Each element $f(x_1, \ldots, x_n)$ of R may naturally be thought of as a $\mathbb{C}$-valued function on X. For S a subset of R, define $V(S) := \{x \in X \mid f(x) = 0\ \forall f \in S\}$. Observe that:

(1) $V(\text{empty set}) = V(\{0\}) = X$ and $V(\{1\}) = \text{empty set}$
(2) $\cap_\alpha V(S_\alpha) = V(\cup_\alpha S_\alpha)$
(3) $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 := \{fg \mid f \in S_1,\ g \in S_2\}$

[1] One possible way to define and think of topologies is via neighbourhoods as the fundamental notion (instead of open sets).

Thus, taking $V(S)$ as S varies over subsets of R to be the closed sets, we get a topology on X. This is called the *Zariski topology*.

For $f \in R$, define $D(f) = \{x \in X \mid f(x) \neq 0\}$. Observe that:

(1) $D(f) = X \setminus V(\{f\})$
(2) $D(f)$ is the empty set if and only if $f = 0$
(3) $X \setminus V(S) = \cup_{f \in S} D(f)$
(4) $D(f) \cap D(g) = D(fg)$

Thus the $D(f)$, as f varies over R, form a basis of open sets for the Zariski topology. Using this and items (2), (4) above, we see that any two non-empty open sets intersect non-trivially, so the Zariski topology is not Hausdorff. Singletons are closed in the Zariski topology, as can be readily verified.

Regularity.

(1) The following conditions are equivalent for a topological space X. The space X is called *regular* if X is Hausdorff and these conditions hold:
  (a) The closed neighbourhoods of a point form a fundamental system of neighbourhoods of that point. (A collection of neighbourhoods of a point is called a *fundamental system* if every neighbourhood of the point contains a neighbourhood from the collection. For example, the open balls $B(x, 1/n) := \{y \mid d(x, y) < 1/n\}$ of radius $1/n$ around a point x in a metric space, as n varies over the positive integers, form a fundamental system of open neighbourhoods of x. The closed balls $\bar{B}(x, 1/n) := \{y \mid d(x, y) \leq 1/n\}$ form a fundamental system of closed neighbourhoods of x.)
  (b) Given a point x and a closed set C in X with $x \notin C$, there exists a neighbourhood U of x such that the closure of U does not intersect C (or, equivalently, there exist disjoint open sets V and W with $x \in V$ and $C \subseteq W$.)
(2) Subspaces of regular spaces are regular. If every point of a topological space has a closed regular neighbourhood, then the space is regular.
(3) A product of regular spaces is regular. Conversely, if a product of non-empty spaces is regular, then so is each factor.

Completely regular or Tychonoff spaces. A topological space X is called *completely regular*, or *Tychonoff*, or $T_{3\frac{1}{2}}$ if it is Hausdorff and the following condition holds: given a closed set C of X and a point x not in C, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

(1) Subspaces of completely regular spaces are completely regular. A product of completely regular spaces is completely regular. If a product of non-empty topological spaces is completely regular, then so is each factor.
(2) Completely regular spaces are regular.
(3) Show that a metric space is completely regular by explicitly constructing a continuous $[0, 1]$-valued function that separates a given closed set from a given point not in it.

Product topology.

(1) Let X_α, with α in some index set I, be a collection of topological spaces. For a map f from a topological space Y to $\prod_\alpha X_\alpha$ (with the product topology) to be continuous it is necessary and sufficient that $p_\alpha \circ f$ be continuous for every α, where p_α are the projections.
(2) A subspace of a countable product of second countable spaces is second countable.

Embedding in cubes. The product space $[0, 1]^I$, where I is an arbitrary index set, is called a *cube*. Every cube is a compact Hausdorff space (compactness follows from Tychonoff's theorem).

(1) Let X be a topological space and $\mathscr{F}$ a collection of continuous functions (possibly not all such functions) from X to the compact interval $[0, 1]$. Let $\varphi : X \to [0, 1]^{\mathscr{F}}$ be the natural map defined by $(\varphi(x))(f) := f(x)$ (recall that $[0, 1]^{\mathscr{F}}$ can be interpreted as the set of functions from $\mathscr{F}$ to $[0, 1]$). Then:
  (a) φ is a continuous function. (This is clear since the projection to any factor of the product gives a continuous function from X to $[0, 1]$ by construction.)
  (b) φ is an injection if $\mathscr{F}$ separates points of X.

(c) φ from X to φX is an open map if $\mathscr{F}$ separates points from closed sets. (Hint: Let U be open in X and u in U. Let $g \in \mathscr{F}$ with $g(u) = a$, $g(X \setminus U) = b$, and $a \neq b$. Let $p_g : [0,1]^{\mathscr{F}} \to [0,1]$ be the projection to the "g-component" (this is just "evaluation at g"). Then $\varphi X \cap p_g^{-1} W$, where W is an open in $[0,1]$ containing a and not b, is an open set in φX containing u and contained in φU.)

(2) A topological space is completely regular if and only if it is the subspace of a compact Hausdorff space. Indeed a completely regular space can be realized as the subspace of a cube by item (1) above: choose $\mathscr{F}$ to be the collection of all continuous $[0,1]$-valued functions.

Normality. A topological space is called *normal* if it is Hausdorff and the following condition holds: given two disjoint closed sets A and B, there exist disjoint neighbourhoods U and V respectively of A and B.

Some sufficient conditions for normality.

(1) A compact Hausdorff space is regular, even normal.
(2) A metric space is normal. (Hint: Let E and F be disjoint closed sets of a metric space X. Put $U = \cup_{e \in E} B(e, \frac{d(e,F)}{2})$, where $d(e,F) := \inf\{d(e,f) \,|\, f \in F\}$, and $B(e, \frac{d(e,F)}{2})$ is the open ball of radius $\frac{d(e,F)}{2}$ around e. Similarly put $V = \cup_{f \in F} B(f, \frac{d(f,E)}{2})$. Then U and V are open neighbourhoods of E and F respectively. Moreover, U and V are disjoint, for the balls $B(e, \frac{d(e,F)}{2})$ and $B(f, \frac{d(f,E)}{2})$ are disjoint for any pair of points e in E and f in F, because if z belongs to their intersection, then $d(e,f) \leq d(e,z) + d(z,f) < \frac{d(e,F)}{2} + \frac{d(f,E)}{2} \leq \max\{d(e,F), d(f,E)\}$, a contradiction.)
(3) Show by example that for two disjoint closed sets A and B in a metric space $d(A,B)$ could be 0.
(4) A Lindelöf regular space is normal. (Recall that a topological space is called *Lindelöf* if every open cover has a countable sub cover.) (Hint: Let A and B be closed sets. For a in A, let U_a be an open set containing A such that $\overline{U_a}$ does not intersect B. Then $\{U_a\}_{a \in A} \cup \{X \setminus A\}$ is an open cover of X. By Lindelöfness of X, we can find a countable subcover $\{U_n\}_{n \geq 1}$ of the cover $\{U_a\}_{a \in A}$ of A. Similarly we can find a countable cover $\{V_n\}_{n \geq 1}$ of B such that each $\overline{V_n}$ is disjoint from A. Now set $U'_n = U_n \setminus \cup_{j \leq n} \overline{V_j}$ and $V'_n = V_n \setminus \cup_{j \leq n} \overline{U_j}$. Then $U = \cup_{n \geq 1} U'_n$ and $V = \cup_{n \geq 1} V'_n$ are disjoint open neighbourhoods of A and B.)
(5) A second countable space is Lindelöf. Thus, by the previous item, a regular second countable space is normal.

Abundance of continuous functions on a normal space: Urysohn's lemma. Urysohn showed that there are enough continuous functions on any normal space to separate disjoint closed subsets: Urysohn's Lemma: given two disjoint closed sets A and B in a normal space X, there exists a continuous function $f : X \to [0,1]$ with $f(A) = 0$ and $f(B) = 1$.

This lemma holds the key to several basic results in the theory, such as his metrization theorem (see below). It follows from it that normal spaces are completely regular. Its proof may be achieved in the following steps:

(1) Let X be a topological space and let D be a dense subset of the non-negative real numbers. Suppose that we have a collection of open sets U_d of X indexed by d in D such that
   (a) $U_d = X$ for $d > 1$ in D, and
   (b) $\overline{U_d} \subseteq U_e$ for $d < e$ in D.

   Define $f : X \to [0,1]$ by $f(x) = \inf\{d \in D \,|\, x \in U_d\}$. Then:
   - For $s \in [0,1]$, we have $f^{-1}[0,s] = \cap_{d \in D, d > s} \overline{U_d}$ and $f^{-1}[0,s) = \cup_{d \in D, d < s} U_d$.
   - f is continuous.

(2) Now let X be normal, and A, B be disjoint closed sets in X. Choose D to be the set of non-negative dyadic rationals. An arbitrary element of $d \neq 0$ in D has the form $(2m+1)/2^n$. We choose open sets U_d in X indexed by $d \in D$ by an induction on n. Put $U_d = X$ for $d > 1$ in D and $U_1 = X \setminus B$. Let U_0 be an open set containing A whose closure does not meet B. Now for a general d, write it as $(2m+1)/2^n$ and take U_d to be an open set such that $\overline{U_{2m/2^n}} \subseteq U_d \subseteq \overline{U_d} \subseteq U_{(2m+2)/2^n}$. Then the conditions (1a) and (1b) of the previous item are met. Now the f as defined in the previous item is continuous with $f(A) = 0$ and $f(B) = 1$.

Metrics.

(1) Let X be a metric space with distance function d. Define the *truncation* d' of d on X by $d'(x,x') := \min\{d(x,x'),1\}$ for x, x' in X. The truncation d' is also a metric. Moreover the topology defined by d' is the same as that defined by d.

(2) Let $\{X_n\}_{n\geq 1}$, be a countable collection of metric spaces. Then the product space $\prod_{n\geq 1} X_n$ is metrizable. Indeed, letting d'_n be the truncation (as defined in item (1) above) of the metric d_n on X_n, we set $d((x_n),(x'_n)) := \sum_{n\geq 1} d'_n(x_n,x'_n)/2^n$. Then d is a metric on $\prod_{n\geq 1} X_n$ whose underlying topology is the product topology.

(3) A topological space is called *separable* if it has a countable dense subset. A second countable space is separable. (Hint: Choose one point from each member of a countable basis. The resulting countable collection of points is dense.) A separable metric space is second countable. (Hint: Suppose that E is a countable dense set of a metric space. Then $\{B(e,\frac{1}{n})\,|\,e\in E, n \text{ positive integer}\}$ is a countable collection of open sets. It is also a basis: given an element u in an open set U, let n be large enough so that $B(u,\frac{1}{n})\subseteq U$ and choose $e\in E$ such that $e\in B(u,\frac{1}{2n})$; then $u\in B(e,\frac{1}{2n})\subseteq B(u,\frac{1}{n})\subseteq U$.)

Urysohn's metrization theorem (characterization of separable metric spaces). The following are equivalent for a topological space X:

(1) It is regular and second countable.
(2) It is a subspace of the product of a countable number of copies of the compact interval $[0,1]$.
(3) It is a separable metric space.

Hint: (1)⇒(2): By item (5) under Sufficient conditions for normality, the space is normal. Now use the result under Embedding in cubes with $\mathscr{F}$ being the countable collection as below. Let $\mathscr{U}$ be a countable basis. Consider the countable set $\mathscr{Z} := \{(U,V)\,|\,U,V\in\mathscr{U} \text{ with } \overline{U}\subseteq V\}$. For each $(U,V)\in\mathscr{Z}$, choose by Urysohn a continuous $[0,1]$-valued function $f_{U,V}$ with $f_{U,V}(\overline{U})=0$ and $f(X\setminus V)=1$. Let $\mathscr{F} := \{f_{U,V}\,|\,(U,V)\in\mathscr{Z}\}$. To observe that $\mathscr{F}$ separates a point x from a closed set C with $x\notin C$, find $(U,V)\in\mathscr{Z}$ with $x\in U$ and $V\subseteq X\setminus B$.

(2)⇒(3): Let $X\subseteq [0,1]^{\text{countable}}$. A countable product of metric spaces can be given a metric that induces the product topology (item (2) under Metrics). A subspace of a metric space being a metric space, it is clear that X is a metric space. A subspace of a countable product of second countable spaces is second countable (item (2) under Product topology), so X is secound countable and so a separable metric space (item (3) under Metrics).

(3)⇒(1): This is easy. Metric spaces are regular: given C closed and $x\notin C$, first observe that $d(x,C)>0$; then $\cup_{c\in C} B(c,\frac{d(x,C)}{2})$ is an open neighbourhood of C which is disjoint from the neighbourhood $B(x,\frac{d(x,C)}{2})$ of x. Separable metric spaces are second countable (item (3) under Metrics).

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