PARTIAL FRACTIONS
A CRITICAL LOOK

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ABSTRACT. Partial fractions are typically introduced in school mathematics textbooks as a method of integration. The integration of rational functions in one variable reduces, by the division algorithm, to that of proper fractions, which are then handled by expressing them as partial fractions. We take a critical look at this method of partial fractions, with special reference to its treatment in the NCERT Class XII Mathematics text.

1. INTRODUCTION [s:intro]

The integration of rational functions reduces, by the division algorithm, to that of proper fractions. Each proper fraction decomposes as a sum of simple proper fractions called *partial fractions*, each of which is easily integrated. This method of partial fractions is the subject of this article, which started off as notes of a lecture addressed to school teachers of mathematics. It inherits from the lecture special focus on the subject material as it appears in §7.3 of the NCERT Mathematics Book for Class XII, which is referred to throughout as the "text".

In the next three sections below, we consider proper fractions of special types: the text restricts itself to cases where the denominator is of degree at most 3. Later on, in §5, we consider the theoretical basis that underlies the method of partial fractions. In particular, we justify the tacit assumptions in the text about the method. The treatment here is comprehensive: it covers all proper fractions and in turn all rational functions.

2. PROPER FRACTIONS WITH DENOMINATOR A PRODUCT OF DISTINCT LINEAR FACTORS [s:x2]

In this section, we consider integration of proper fractions where the denominator is a product of distinct linear factors.

2.1. The text's solution.

As a simple example of this type, let us consider

$$\text{Find} \quad \int \frac{1}{x^2-9}\,dx \tag{1}$$

We first apply to the above example the text's suggested method of solution. Begin by factoring: $x^2-9=(x+3)(x-3)$. Put

$$\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3} \tag{2}$$

This article started off as notes of a lecture given on 30 September 2013 at a workshop at IMSc for school mathematics teachers. Corrections, comments, suggestions for improvement, criticisms, etc. are welcome. Please send them to the author at `knr.imsc@gmail.com` or `knr@imsc.res.in`.

where A and B are unknowns to be determined. Multiplying both sides of the equation by $(x+3)(x-3)$, we get

$$1 = A(x+3) + B(x-3) \tag{3}$$

Equating the coefficients of x and the constant terms on both sides, we get:

$$\begin{aligned} A \;+\; B \;&=\; 0 \\ 3A - 3B \;&=\; 1 \end{aligned} \tag{4}$$

which we solve to find A and B. Multiplying the first equation by 3 and adding the result to the second, we get $6A = 1$ or $A = 1/6$. Plugging this into the first, we get $B = -1/6$. Substituting these into (2) and integrating, we get:

$$\begin{aligned} \int \frac{1}{x^2-9}\,dx \;=\; \int \frac{1}{(x+3)(x-3)}\,dx \;&=\; \int \frac{1}{6(x+3)}\,dx \;-\; \int \frac{1}{6(x-3)}\,dx \\ &=\; \frac{1}{6}\log|x+3| \;-\; \frac{1}{6}\log|x-3| \;+\; \text{constant} \end{aligned} \tag{5}$$

[ss:t1:altsol]

2.2. **An alternative method.** We now suggest an alternative, quick, way to determine A and B in (3), with which perhaps the reader is already familiar. Putting $x=3$, we get $1 = A\cdot 6$, so $A = 1/6$; putting $x=-3$, we get $1 = B\cdot(-6)$, so $B = -1/6$.

The advantage of the above method over that of equating coefficients becomes more pronounced when the denominator has three or more distinct factors, as for instance in:

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \tag{6}$$

Multiplying through by $(x-1)(x-2)(x-3)$, we get

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \tag{7}$$

To determine A, B, and C, we need only set x equal to 1, 2, and 3:

$$1 = A(-1)(-2) \text{ so } A = 1/2; \quad 2 = B(1)(-1) \text{ so } B = -2; \quad 3 = C(2)(1) \text{ so } C = 3/2. \tag{8}$$

2.3. **Closed form expression in general for the coefficients.** The alternative method is powerful enough to let us fearlessly tackle the general case when the denominator is a product of an arbitrary number of distinct linear factors. In fact, we can write down closed form formulas for the coefficients in the expression of the integrand as partial fractions. We illustrate how to do this when the denominator is a product of four distinct linear factors. The restriction to four factors is only for the sake of notational simplicity.

Let $f(x)$ be a polynomial of degree at most 3. Let a, b, c, d be distinct numbers. Wanting to integrate $f(x)/(x-a)(x-b)(x-c)(x-d)$, we put[1]

$$\frac{f(x)}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} \tag{9}$$

[1]The justification for (2), (6), (9), and all such subsequent assumptions is provided by (73).

Multiplying through by $(x-a)(x-b)(x-c)(x-d)$, and setting successively x equal to a, b, c, d, we get:

$$\begin{aligned} A &= \frac{f(a)}{(a-b)(a-c)(a-d)} & B &= \frac{f(b)}{(b-a)(b-c)(b-d)} \\ C &= \frac{f(c)}{(c-a)(c-b)(c-d)} & D &= \frac{f(d)}{(d-a)(d-b)(d-c)} \end{aligned} \tag{10}$$

It is time now to pose the question:

$$\text{Does the method of equating coefficients have any advantage at all over the alternative method of plugging values in for } x? \tag{11}$$

See Exercise 1 for an answer.

[ss:li]

2.4. **Lagrange interpolation.** The underlying idea in the alternative method above also underlies the method of LAGRANGE INTERPOLATION which we now describe. Let $f(x)$ be a linear function of x:

$$f(x) = \alpha x + \beta \tag{12}$$

where α and β are constants. The graph of $f(x)$ is a line. Suppose we know that the line passes through two points (a, u) and (b, v) where $a \neq b$. There being always one and only such line, it is clear that if we specify the values $f(a)$ and $f(b)$ at a and b of $f(x)$, then the function $f(x)$ is determined.

In a similar way, for any positive integer m, there is one and only one polynomial $f(x)$ of degree less than m with specified values at m distinct values of x. See Exercise 1 for a proof. The question now is: how to determine the polynomial given the values? For example, which polynomial of degree at most 2 has values 2, 1, and 4 at 1, 2, and 4 respectively?

Lagrange interpolation is a method to write the polynomial down. Put

$$f(x) = A(x-2)(x-4) + B(x-1)(x-4) + C(x-1)(x-2) \tag{13}$$

where A, B, and C are constants to be determined. Clearly $f(x)$ is a polynomial of degree at most 2. The constants A, B, and C are now readily determined by plugging in successively x equal to 1, 2, 4 and respectively $f(x)$ equal to 2, 1, 4:

$$2 = A(-1)(-3), \text{ so } A = 2/3;\ 1 = B(1)(-2), \text{ so } B = -1/2;\ 4 = C(3)(2), \text{ so } C = 2/3 \tag{14}$$

Plugging these values of A, B, and C into (13) and simplifying, we get

$$f(x) = \frac{5}{6}x^2 - \frac{7}{2}x + \frac{14}{3} \tag{15}$$

Let us now consider a polynomial $f(x)$ of degree at most 3 whose values $f(a)$, $f(b)$, $f(c)$, and $f(d)$ at four distinct values a, b, c, and d are specified. Put

$$\begin{aligned} f(x) \ = \ & A(x-b)(x-c)(x-d) \ + \ B(x-a)(x-c)(x-d) \\ & + C(x-a)(x-b)(x-d) \ + \ D(x-a)(x-b)(x-c) \end{aligned} \tag{16}$$

Plugging in successively the values a, b, c, d for x and corresponding values for $f(a)$, $f(b)$, $f(c)$, $f(d)$ for $f(x)$, we see that A, B, C, D are given by the formulas (10).

[s:x2]

3. Fractions with denominator a power of a linear form

In this section, we consider integration of rational functions where the denominator is of the form $(x-a)^n$. We consider only proper fractions first, but in §3.6 there is no such restriction.

3.1. The text's method of solution. As a simple example of this type, consider

$$\int \frac{3x-1}{(x+2)^2}dx \tag{17}$$

Let us first follow the suggested method of solution in the text. Put

$$\frac{3x-1}{(x+2)^2} = \frac{A_1}{(x+2)} + \frac{A_2}{(x+2)^2} \tag{18}$$

Multiply through by $(x+2)^2$:

$$3x-1 = A_1(x+2) + A_2 \tag{19}$$

Equate the coefficient of x and the constant term on both sides:

$$3 = A_1 \qquad -1 = 2A_1 + A_2 \tag{20}$$

Solve these for A_1 and A_2 and substitute back into (18):

$$\frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} + \frac{-7}{(x+2)^2} \tag{21}$$

Finally, integrate:

$$\int \frac{3x-1}{(x+2)^2}dx = \int \frac{3}{(x+2)}dx + \int \frac{-7}{(x+2)^2}dx = 3\log|x+2| + \frac{7}{x+2} + \text{constant} \tag{22}$$

3.2. Comments on the text's method of solution. The text's procedure above generalizes to the case when the denominator is a higher power, for instance, $(x+2)^6$. Indeed, for $f(x)$ a polynomial of degree at most 5, we put

$$\frac{f(x)}{(x+2)^6} = \frac{A_1}{(x+2)} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} + \frac{A_4}{(x+2)^4} + \frac{A_5}{(x+2)^5} + \frac{A_6}{(x+2)^6} \tag{23}$$

Multiply through by $(x+2)^6$:

$$f(x) = A_1(x+2)^5 + A_2(x+2)^4 + A_3(x+2)^3 + A_4(x+2)^2 + A_5(x+2) + A_6 \tag{24}$$

Looking to equate coefficients of powers of x on both sides, we consider the coefficients of x^5, x^4, x^3, x^2, x^1, x^0 on the right side. Using the binomial theorem to expand the powers of

$(x+2)$, we see that these coefficients are, respectively:

$$\begin{aligned}
&A_1\\
&\binom{5}{1}2^1A_1 + A_2\\
&\binom{5}{2}2^2A_1 + \binom{4}{1}2^1A_2 + A_3\\
&\binom{5}{3}2^3A_1 + \binom{4}{2}2^2A_3 + \binom{3}{1}2^1A_1 + A_4\\
&\binom{5}{4}2^4A_1 + \binom{4}{3}2^3A_2 + \binom{3}{2}2^2A_3 + \binom{2}{1}2^1A_4 + A_5\\
&\binom{5}{5}2^5A_1 + \binom{4}{4}2^4A_2 + \binom{3}{3}2^3A_3 + \binom{2}{2}2^2A_4 + \binom{1}{1}2^1A_5 + A_6
\end{aligned} \tag{25}$$

While these expressions get increasingly complicated, there is a pattern in them. Namely, the first expression involves only A_1, the second only A_1, A_2, the third only A_1, A_2, A_3, etc. Moreover, the coefficient of A_1 in the first, of A_2 in the second, of A_3 in the third, etc. are all 1.

It is clear how to solve the system of linear equations obtained from equating coefficients of powers of x: the first equation—the one obtained by equating coefficients of x^6—gives A_1 directly; substituting this value into the second—the one for coefficients of x^5—we solve for A_2; substituting the values of A_1 and A_2 into the third, we solve for A_3; and so on.

Example 3.2.1. If the degree of $f(x)$ is small, then the linear system above is especially simple and readily solved. For example, if $f(x) = 5x+3$, then it is clear—equating successively the coefficients of x^5, x^4, x^3, x^2—that A_1, A_2, A_3, A_4 vanish. Equating the coefficients of x, we get $A_5 = 5$. Equating the constant terms, we get $2A_5 + A_6 = 3$, so $A_6 = 3 - 2\cdot 5 = -7$. We have:

$$\int \frac{5x+3}{(x+2)^6}dx = \int \frac{5}{(x+2)^5}dx + \int \frac{-7}{(x+2)^6}dx = \frac{-5}{4(x+2)^4} + \frac{7}{5(x+2)^5} + \text{constant} \tag{26}$$

[e:long]

Example 3.2.2. If however the degree of $f(x)$ is large—note that it can at most be 5 for the fraction $f(x)/(x+2)^6$ to be proper—e.g., if $f(x) = x^5+2x^4+3x+1$, then the procedure, although still within reach of calculation by hand, is much easier described than carried out! We will soon return to this example (see Example 3.3.1 below), where we try out on it the alternative procedure described in the next subsection for finding the coefficients A_i. The reader is invited to compare the levels of difficulty of carrying out the two procedures.

3.3. **Alternative method of solution.** We now look at an alternative way of determining the A_i in (24), the idea behind which is far-reaching. Recall that our alternative solution in Examples of type 1 (see §2.2) consisted of plugging in various values for x. These values were precisely the zeros of the various linear factors appearing in the denominator of the original given fraction (to be integrated). There were as many of these values as there were coefficients to be determined, and we were able to determine all the coefficients directly by plugging in these values.

Here there is only one such value, namely $x = -2$. Substituting this into (24), we get A_6:

$$A_6 = f(-2). \tag{27}$$

But what do we do next? While we may plug in any value of x of our choice, no other value especially suggests itself. So, how do we proceed? Here is the idea:

$$\boxed{\text{differentiate (24) and then substitute } x = -2} \tag{28}$$

Differentiating (24), we get

$$f'(x) = 5A_1(x+2)^4 + 4A_2(x+2)^3 + 3A_3(x+2)^2 + 2A_4(x+2) + A_5 \qquad \text{where } f'(x) = \frac{df}{dx} \tag{29}$$

Substituting $x = -2$ into the above, we get

$$A_5 = f'(-2) \tag{30}$$

And we may $\boxed{\text{repeat the process}}$. Differentiting (29), we get

$$f''(x) = 4 \cdot 5A_1(x+2)^3 + 3 \cdot 4A_2(x+2)^2 + 2 \cdot 3A_3 \cdot (x+2) + 2A_4 \tag{31}$$

where $f''(x) = df'(x)/dx = d^2 f(x)/dx^2$. Substituting $x = -2$ into the above, we get

$$A_4 = \frac{f''(-2)}{2} \tag{32}$$

By successive differentiations and evaluations, we get:

$$A_3 = \frac{f^{(3)}(-2)}{3!} \qquad A_2 = \frac{f^{(4)}(-2)}{4!} \qquad A_1 = \frac{f^{(5)}(-2)}{5!} \tag{33}$$

where $f^{(m)}(x)$ stands for the m^{th} derivative $d^m f/dx^m$ of the polynomial $f(x)$. Note the pattern of progression in (27), (30), (32), and (33). Substituting these into (24), we have:

$$\begin{aligned} f(x) \;=\; & \frac{f^{(5)}(-2)}{5!}(x+2)^5 \;+\; \frac{f^{(4)}(-2)}{4!}(x+2)^4 \;+\; \frac{f^{(3)}(-2)}{3!}(x+2)^3 \;+ \\ & \frac{f^{(2)}(-2)}{2!}(x+2)^2 \;+\; \frac{f^{(1)}(-2)}{1!}(x+2) \;+\; f(-2) \end{aligned} \tag{34}$$

We have thus determined the coefficients A_1, A_2, A_3, A_4, and A_5 in (24).

There are several comments we would like to make about (34). But first let's look at Example 3.2.2 in its light.

[e:long:alt]

Example 3.3.1. Consider, as in Example 3.2.2, the problem of integrating $f(x)/(x+2)^6$ where $f(x) = x^5 + 2x^4 + 3x + 1$. We apply (34) to write $f(x)$ in the form (24). An easy calculation gives us:

$$\begin{gathered} f'(x) = 5x^4 + 8x^3 + 3 \qquad \frac{f^{(2)}(x)}{2!} = 10x^3 + 12x^2 \\ \frac{f^{(3)}(x)}{3!} = 10x^2 + 8x \qquad \frac{f^{(4)}(x)}{4!} = 5x + 2 \qquad \frac{f^{(5)}(x)}{5!} = 1 \end{gathered} \tag{35}$$

Plugging in $x = -2$, we get[2]

$$f(-2) = -5,\ f'(-2) = 19,\ \frac{f^2(-2)}{2!} = -32,\ \frac{f^3(-2)}{3!} = 24,\ \frac{f^4(-2)}{4!} = -8,\ \frac{f^5(-2)}{5!} = 1 \quad (36)$$

Applying (34), we get:

$$f(x) = x^5+2x^4+3x+1 = (x+2)^5-8(x+2)^4+24(x+2)^3-32(x+2)^2+19(x+2)-5 \quad (37)$$

Dividing through by $(x+2)^6$, we get:

$$\frac{f(x)}{(x+2)^6} = \frac{1}{(x+2)} - \frac{8}{(x+2)^2} + \frac{24}{(x+2)^3} - \frac{32}{(x+2)^4} + \frac{19}{(x+2)^5} - \frac{5}{(x+2)^6} \quad (38)$$

Finally, integrating term by term, we get:

$$\int \frac{f(x)}{(x+2)^6}dx = \log|x+2| + \frac{8}{(x+2)} - \frac{12}{(x+2)^2} + \frac{32}{3(x+2)^3} - \frac{19}{4(x+2)^4} + \frac{1}{(x+2)^5} + C \quad (39)$$

The advantage over the text's method of the alternative one is perhaps now clear. In analogy with (11) let us ask

Does the method of equating coefficients have any advantage at all over the alternative method of repeatedly differentiating and evaluating? (40)

See Exercise 4 for an answer.

[ss:consolidate

3.4. **More on the idea behind the alternative solution: Taylor series.** As already remarked, the idea behind the the alternative solution is far-reaching. In the remaining subsections of this section, we will explore the idea a little more. As the reader may guess, what we will see is only the tip of the iceberg.[3]

Following the same steps as in the alternative solution, we can establish the following: see Exercise 2. Given a polynomial $f(x)$ of degree m and a number a, we have:

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(m)}(a)}{m!}(x-a)^m \quad (41)$$

We may remove the reference to the degree of the polynomial $f(x)$ and write

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \quad (42)$$

If $f(x)$ has degree m, then derivatives of $f(x)$ of order higher than m are identically zero, so (42) reduces to (41). Thus (42) is justified. Although its right hand side looks like an infinite sum, it is a finite one for any specific polynomial.

Equation (42), or its more succinct version

$$f(x) \ = \ \sum_{m\geq 0} \frac{f^{(m)}(a)}{m!}(x-a)^m \quad (43)$$

is called the TAYLOR SERIES EXPANSION of $f(x)$ at a.

[ss:ecossin]

3.5. **Taylor series expansions of exponential, sine, and cosine functions.** Taylor series expansions (43) can be made sense of for a large class of functions of which polynomials are but a small subset. We now give some examples of such expansions.[4] Unlike in the case of polynomials, the right hand side of (43) is not a finite sum in these examples, and so needs to be appropriately interpreted.

3.5.1. *Geometric series.* Put $f(x) = 1/(1-x)$ and $a = 0$. As an easy calculation shows, $f'(x) = 1/(1-x)^2$, $f''(x)/2! = 1/(1-x)^3$, $\ldots$, $f^{(m)}(x)/m! = 1/(1-x)^{m+1}$. So all the derivatives are 1 when evaluated at 0, and (43) becomes

$$\frac{1}{1-x} \;=\; 1 \;+\; x \;+\; x^2 \;+\; x^3 \;+\; \cdots \tag{44}$$

The reader is perhaps familiar with the sum of the geometric series:

$$\frac{1-x^r}{1-x} \;=\; 1 \;+\; x \;+\; x^2 \;+\; \cdots \;+\; x^r \tag{45}$$

Equation (44) may be viewed as the limit in case $|x| < 1$ of (45) as r tends to infinity.[5]

3.5.2. *Exponential series.* Put $f(x) = e^x$ and $a = 0$. Then, for all $m \geq 0$, $f^{(m)}(x) = e^x$, so $f^{(m)}(0) = 1$. Thus (43) becomes the famous exponential series with which the reader is perhaps familiar:

$$e^x \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \cdots \tag{46}$$

3.5.3. *Sine series.* Put $f(x) = \sin x$ and $a = 0$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and the pattern repeats:

> $f^{(m)}(x)$ is $\sin x$, $\cos x$, $-\sin x$, or $-\cos x$, accordingly as m leaves remainder 0, 1, 2, or 3 on division by 4.

We have

$$f^{(m)}(0) = \begin{cases} 0 & \text{if } m \text{ is even} \\ (-1)^{(m-1)/2} & \text{if } m \text{ is odd} \end{cases} \tag{47}$$

Thus the Taylor series expansion of $\sin x$ at 0 is:

$$\sin x \;=\; x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\; \frac{x^9}{9!} \;-\; \frac{x^{11}}{11!} \;+\; - \;\cdots \tag{48}$$

3.5.4. *Cosine series.* By calculations similar to those above for the sine series, we get the Taylor series expansion of $\cos x$ at 0:

$$\cos x \;=\; 1 \;-\; \frac{x^2}{2!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^6}{6!} \;+\; \frac{x^8}{8!} \;-\; \frac{x^{10}}{10!} \;+\; - \;\cdots \tag{49}$$

[ss:t2:gensol]

3.6. **The general solution to Examples of type 2.** Using the Taylor expansion (43), we can even write a closed form expression for the integral of $f(x)/(x-a)^n$ where $f(x)$ is any polynomial. In this subsection, $f(x)$ may have any degree: in other words, the fraction $f(x)/(x-a)^n$ need not be proper. All sums in the next two equations are actually finite. Indeed, for any polynomial $f(x)$, all derivatives of orders larger than its degree vanish.

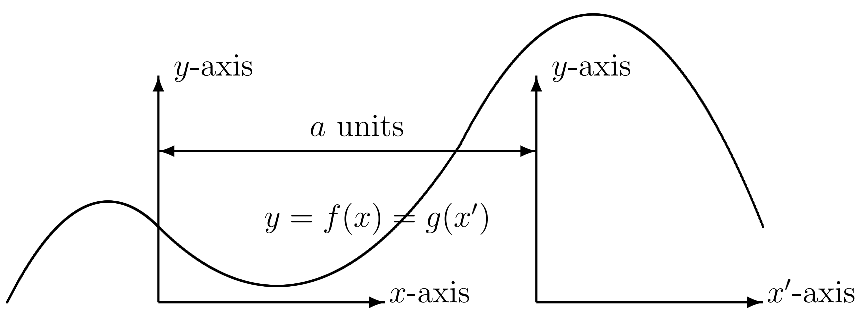
By the Taylor expansion (43) of $f(x)$, we have

$$\frac{f(x)}{(x-a)^n} = \sum_{m\geq 0} \frac{(f^{(m)}(a)/m!)(x-a)^m}{(x-a)^n} = \sum_{m\geq 0} \frac{f^{(m)}(a)}{m!}(x-a)^{m-n} \tag{50}$$

The integral of $(x-a)^{m-n}$ being, up to addition of a constant, $(x-a)^{m-n+1}/(m-n+1)$ except when $m-n=-1$, in which case it is $\log|x-a|$, we get:

$$\int \frac{f(x)}{(x-a)^n}dx \;=\; \frac{f^{(n-1)}(a)}{(n-1)!}\log|x-a| \;+ \sum_{m\geq 0,\ m\neq n-1} \frac{f^{(m)}(a)}{m!}\frac{(x-a)^{m-n+1}}{(m-n+1)} \;+\; \text{constant} \tag{51}$$

3.7. **Shifting of axis and the Taylor expansion.** Choice of an appropriate co-ordinate system can be important in problem solving. Suppose that we need to shift the y-axis as in: [ss:graph]



We have, as a result of the shift, $x' = x - a$. Suppose that $y = f(x)$ is given explicitly, for example, $y = x^5 - 3x^4 + 8x + 11$, and that we want to find the polynomial g such that $y = g(x')$, where a is say 2. One way to do this would be to substitute $x = x' + 2$ into the formula for y to get $y = (x'+2)^5 - 3(x'+2)^4 + 8(x'+2) + 11$. We could then expand using the binomial formula and thereby would have expressed y as a polynomial in x'.

An alternative (and perhaps better) way to accomplish the calculation is to expand the given function $y = f(x)$ in Taylor series (41) around $a = 2$. To this end, we compute the derivatives of orders up to 5 at 2 of $y = f(x)$:

$$f'(x) = 5x^4 - 12x^3 + 8,\ \ \frac{f''(x)}{2!} = 10x^3 - 18x^2,\ \ \frac{f^{(3)}}{3!} = 10x^2 - 12x,\ \ \frac{f^{(4)}}{4!} = 5x - 3,\ \ \frac{f^{(5)}}{5!} = 1$$

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[2] Observe that these values are all integers. Is this a coincidence? See Exercise 5.

[3] The iceberg in this case is enormous.

[4] In all the examples below, $a = 0$. Taylor series with $a = 0$ are also referred to as Maclaurin series.

[5] For more on this matter, one may refer to the talk by Professor D. S. Ramana in this workshop.

Plugging $x = 2$ we get:

$$f(2) = 11, \quad f'(x) = -8, \quad \frac{f''(x)}{2!} = 8, \quad \frac{f^{(3)}}{3!} = 16, \quad \frac{f^{(4)}}{4!} = 7, \quad \frac{f^{(5)}}{5!} = 1$$

Thus we have

$$\begin{aligned} y = f(x) &= 11 - 8(x-2) + 8(x-2)^2 + 16(x-2)^3 + 7(x-2)^4 + (x-2)^5 \\ &= \boxed{11 - 8x' + 8x'^2 + 16x'^3 + 7x'^4 + x'^5} \end{aligned}$$

[ss:dva]

3.8. **Rectilinear motion with constant acceleration.** The reader is in all likelihood familiar with the following formula from kinematics:

$$s = s_0 + ut + \frac{1}{2}at^2 \tag{52}$$

Here t denotes time and s the displacement of a point particle as measured from a given point of reference. The particle is assumed to move along a straight line with constant acceleration a, as for example a freely falling stone that is dropped or thrown upwards or downwards from a tower, where we ignore altogether the effect of air resistance on the falling stone; s_0 and u are the initial values of displacement and velocity.

Equation (52) is just the Taylor expansion (43) at time $t = 0$ for the displacement function s. Indeed, velocity and acceleration are by definition just the first and second derivatives of displacement with respect to time. Thus s_0, u, and a are just the values at time $t = 0$ of s and its first two derivatives. The assumption about constant acceleration means that all derivatives of orders 3 or higher vanish.

[s:x3]

4. Proper fractions of degree at most 3

In a proper fraction $N(x)/D(x)$, we may assume $D(x)$ is monic, by dividing both numerator and denominator by the leading coefficient of $D(x)$. We suppose in this section that $D(x)$ is monic of degree at most 3.

If $D(x)$ is of degree 1, then it is of the form $x - a$; if it is of degree 2, then it is of one of three forms:

$$(x-a)(x-b) \text{ with } a \neq b, \qquad (x-a)^2, \qquad \text{or} \quad x^2 + 2bx + c \text{ with } b^2 - c < 0 \tag{53}$$

Suppose $D(x)$ has degree 3. Then it has at least one linear factor, say $x - a$. The quotient by this linear factor being of degree 2 has one of three forms above, and so there are four possibilities for $D(x)$ in this case:

$$\begin{gathered} (x-a)(x-b)(x-c) \text{ with } a,\ b,\ c \text{ distinct}, \qquad (x-a)^2(x-b) \text{ with } a \neq b, \\ (x-a)^3, \qquad (x-a)(x^2+2bx+c) \text{ with } b^2 - c < 0 \end{gathered} \tag{54}$$

Out of the eight forms above, three are not treated by the earlier sections: namely, the last one of (53), the second and fourth of (54). We treat these in turn in the subsections below.

[ss:Q]

4.1. **Proper fraction with denominator $x^2+2bx+c$, $b^2-c<0$.** Complete the square to write

$$x^2+2bx+c=(x+b)^2+(c-b^2)=E(y^2+1) \quad \text{where } y=\frac{x+b}{\sqrt{E}} \text{ and } E=c-b^2 \tag{55}$$

We have

$$\begin{aligned}\int \frac{fx+g}{x^2+2bx+c}\,dx &= \int \frac{f(\sqrt{E}y-b)+g}{E(y^2+1)}\sqrt{E}\,dy \\ &= \frac{f}{2}\int \frac{2y}{y^2+1}\,dy \;+\; \frac{-bf+g}{\sqrt{E}}\int \frac{dy}{y^2+1} \\ &= \frac{f}{2}\log\,(y^2+1) \;+\; \frac{-bf+g}{\sqrt{E}}\tan^{-1}y \;+\; \text{constant} \\ &= \frac{f}{2}\log\,(\frac{x^2+2bx+c}{c-b^2}) \;+\; \frac{-bf+g}{\sqrt{c-b^2}}\tan^{-1}\frac{x+b}{\sqrt{c-b^2}} \;+\; \text{constant}\end{aligned} \tag{56}$$

4.2. **Proper fraction with denominator $(x-a)^2(x-b)$, $a\neq b$.** Let $f(x)$ be a polynomial of degree at most 2. Put [ss:a2b]

$$\frac{f(x)}{(x-a)^2(x-b)} \;=\; \frac{A_1}{x-a} \;+\; \frac{A_2}{(x-a)^2} \;+\; \frac{B}{x-b} \tag{57}$$

Let us solve for A_1, A_2, and B in a fashion that generalizes to the case when the denominator is a product of several repeated linear factors: see Exercise 7. First multiply through by $(x-b)$ to get:

$$\frac{f(x)}{(x-a)^2} \;=\; \frac{A_1(x-b)}{(x-a)} \;+\; \frac{A_2(x-b)}{(x-a)^2} \;+\; B \tag{58}$$

Put $x=b$ to get $B=f(b)/(b-a)^2$. Now, multiply (57) through by $(x-a)^2$ to get

$$\frac{f(x)}{x-b} \;=\; A_1(x-a) \;+\; A_2 \;+\; \frac{B(x-a)^2}{x-b} \tag{59}$$

Put $x=a$ to get $A_2=f(a)/(a-b)$. Now differentiate (63) and put $x=a$ to get $A_1=(f(x)/(x-b))'(a)=(f'(a)(a-b)-f(a))/(a-b)^2$. Thus

$$\begin{aligned}\int \frac{f(x)}{(x-a)^2(x-b)}\,dx \quad &= \quad \frac{f'(a)(a-b)-f(a)}{(a-b)^2}\log|x-a| \\ &\quad - \frac{f(a)}{(a-b)}\frac{1}{(x-a)} \;+\; \frac{f(b)}{(b-a)^2}\log|x-b| \;+\; \text{constant}\end{aligned} \tag{60}$$

4.3. **Proper fraction with denominator $(x-a)(x^2+2bx+c)$, $b^2-c<0$.** Let $f(x)$ be a quadratic polynomial of degree at most 2, and set $Q(x)=x^2+2bx+c$ (for notational convenience). Put [ss:aQ]

$$\frac{f(x)}{(x-a)Q(x)} \;=\; \frac{A}{x-a} \;+\; \frac{Bx+C}{Q(x)} \tag{61}$$

Once we determine A, B, and C, we are done. Indeed, the first term on the right is easy to integrate, and we know the integral of the second term from §4.1: we need only substitute the values of B, C respectively for f, g in (56).

First multiply (62) through by $(x-a)$ to get:

$$\frac{f(x)}{Q(x)} \;=\; A \;+\; \frac{Bx+C}{Q(x)}(x-a) \tag{62}$$

Put $x=a$ in the above to get $A = f(a)/Q(a)$.

Now multiply (62) through by $Q(x)$:

$$\frac{f(x)}{x-a} \;=\; \frac{A\,Q(x)}{x-a} \;+\; Bx+C \tag{63}$$

Thus $Bx+C$ is the quotient on division by $x-a$ of $f(x)-AQ(x) = f(x)-f(a)Q(x)/Q(a)$. Expanding $f(x)-f(a)Q(x)/Q(a)$ by (41):

$$f(x) - \frac{f(a)}{Q(a)}Q(x) \;=\; (f'(a) - \frac{f(a)}{Q(a)}Q'(a))(x-a) \;+\; \frac{1}{2}(f''(a) - \frac{f(a)}{Q(a)}Q''(a))(x-a)^2 \tag{64}$$

Equating $Bx+C = B(x-a)+(aB+C)$ with the quotient of the right hand side of the above equation by $x-a$:

$$\begin{aligned} B \;&=\; \frac{1}{2}(f''(a) - \frac{f(a)}{Q(a)}Q''(a)) \;=\; \frac{f''(a)}{2} \;-\; \frac{f(a)}{Q(a)}, \\ C \;&=\; -aB \;+\; (f'(a) - \frac{f(a)}{Q(a)}Q'(a)) \;=\; f'(a) \;-\; \frac{f(a)Q'(a)}{Q(a)} \;-\; \frac{af''(x)}{2} \;+\; \frac{af(a)}{Q(a)} \end{aligned} \tag{65}$$

[s:general]

5. Integration of rational functions in general

Here we look at the theoretical basis for the method of partial fractions. Let $N(x)/D(x)$ be an arbitrary rational function. This means that $N(x)$ and $D(x)$ are polynomials with real coefficients in a single variable x, and $D(x)\neq 0$. Dividing both $N(x)$ and $D(x)$ by the leading coefficient of $D(x)$, we may assume that $D(x)$ is *monic*, that is, its leading coefficient is 1.

Our goal is to integrate $N(x)/D(x)$, that is, to find its anti-derivative. We reach it in three steps, described respectively in the three subsections below. First we reduce to the case when the the rational function is a proper fraction, that is, when the degree of $N(x)$ is less than that of $D(x)$. Then we show that an arbitrary proper fraction has an expression as a sum of certain special types of proper fractions, called *partial fractions*. Finally, we integrate all partial fractions.

[ss:divalg]

5.1. **Reduction to the case of proper fraction: division algorithm.** The Division algorithm for polynomials states:

> Given polynomials $N(x)$ and $D(x)$, with $D(x)\neq 0$, there exist unique polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < \deg D(x)$ such that
>
> $$N(x) = Q(x)D(x)+R(x) \qquad \text{or, equivalently} \quad \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \tag{66}$$

The polynomials $Q(x)$ and $R(x)$ are found by the process of "long division".

Given a rational function $N(x)/D(x)$ to be integrated, we write it in the form (66). Now, $Q(x)$ being a polynomial, we know how to integrate it, so it suffices to integrate $R(x)/D(x)$. Thus we are reduced to the integration of proper fractions. We will see what to do with proper fractions in §5.3. Until then, we take a long detour in which we prove results that we will need. These results being interesting in their own right, we hope not to lose the reader on the detour.

5.1.1. Corollaries of division algorithm; Euclidean algorithm. In the rest of this subsection, we record some corollaries of the division algorithm that are used in what follows, not always explicitly:

Corollary 5.1.1. (Remainder theorem) *A number a is the root of a polynomial $P(x)$* (which by definition means $P(a) = 0$) *if and only if $x - a$ divides $P(x)$.*

Corollary 5.1.2. *If a complex number α is a root of a polynomial $P(x)$ with real coefficients, then so is its complex conjugate $\bar{\alpha}$.*

Proof. Write $P(x) = (x - \alpha)Q(x)$. Taking complex conjugates of coeffiecients in this equation, we get $P(x) = (x - \bar{\alpha})\bar{Q}(x)$, so $\bar{\alpha}$ is also a root of $P(x)$. □

Let $K(x)$ and $L(x)$ be polynomials. We say that a monic polynomial $G(x)$ is the *greatest common divisor*, GCD for short, of $K(x)$ and $L(x)$ if it divides both of them and is divisible by any polynomial that divides both of them.

Corollary 5.1.3. (Euclidean algorithm for the GCD) *GCDs always exist. They may be found by the Euclidean algorithm as in the case of integers. If $G(x)$ is the GCD of $K(x)$ and $L(x)$, then there exist polynomials $A(x)$ and $B(x)$ such that $A(x)K(x) + B(x)L(x) = G(x)$.*

Two polynomials are said to be *coprime* if their GCD is 1. If $K(x)$ and $L(x)$ are relatively prime, then there exist $A(x)$ and $B(x)$ such that $A(x)K(x) + B(x)L(x) = 1$. One may find $A(x)$ and $B(x)$ from the Euclidean algorithm.

Corollary 5.1.4. *If $K(x)$ and $L(x)$ are coprime and both divide $P(x)$, then their product $K(x)L(x)$ divides $P(x)$.*

Proof. Find $A(x)$ and $B(x)$ such that $A(x)K(x) + B(x)L(x) = 1$. Multiplying by $P(x)$, we get $P(x)A(x)K(x) + P(x)B(x)L(x) = P(x)$. Since $L(x)$ divides $P(x)$, the first term on the left is divisible by $K(x)L(x)$. Similarly, since $K(x)$ divides $P(x)$, the second term on the left is divisible by $K(x)L(x)$. Thus $K(x)L(x)$ also divides the right hand side. □

Corollary 5.1.5. *If $K(x)$ and $L(x)$ are coprime and $K(x)$ divides $P(x)L(x)$, then $K(x)$ divides $P(x)$.*

Proof. Find $A(x)$ and $B(x)$ such that $A(x)K(x) + B(x)L(x) = 1$. Multiplying by $P(x)$, we get $P(x)A(x)K(x) + P(x)B(x)L(x) = P(x)$. The first term on the left is evidently divisible by $K(x)L(x)$. Similarly, since $K(x)$ divides $P(x)L(x)$, the second term on the left is also divisible by $K(x)$. Thus $K(x)$ divides the right hand side. □

[ss:crt]

5.2. **Proper fractions as sums of partial fractions: Chinese Remainder Theorem.** The next theorem is the analogue for polynomials with real coefficients of the familiar prime factorization of integers.

[t:factor]

Theorem 5.2.1. *Every monic polynomial $D(x)$ with real coefficients has a unique expression, up to reordering of the factors, as a product in the following form:*

$$D(x) = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} Q_1^{\ell_1} \cdots Q_s^{\ell_s} \tag{67}$$

where the $a_1, \ldots, a_r$ are distinct reals, the Q_j are distinct polynomials each of the form $x^2 + b_j x + c_j$ with $b_j^2 - 4c_j < 0$, and the k_i, ℓ_j are positive integers.[6] *Here $k_1 + \cdots + k_r + 2\ell_1 + \cdots + \ell_s = \deg D(x)$.*

Proof. To prove existence, proceed by induction on the degree of $D(x)$. If it is of degree 1 then is of the form $x - a$ and we are done. Suppose that $D(x)$ has degree at least 2. If $D(x)$ has a real root a, then $D(x)$ is divisible by $x - a$, and we get a factorization (67) of $D(x)$ from that of the quotient $D(x)/(x - a)$ by multiplication by $x - a$.

Suppose that $D(x)$ does not have a real root. By the fundamental theorem of algebra, there is a non-real complex root, say α, of $D(x)$. By Corollary 5.1.2, $\bar{\alpha}$ is then also a root of $D(x)$. Since $x - \alpha$ and $x - \bar{\alpha}$ are relatively prime, both divide $D(x)$. Put $Q(x) = (x - \alpha)(x - \bar{\alpha})$. Then $Q = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$. Then $(\alpha + \bar{\alpha})^2 - 4\alpha\bar{\alpha} = (\alpha - \bar{\alpha})^2 < 0$, because $\alpha - \bar{\alpha}$ is purely imagainary and non-zero. Now the factorization (67) for $D(x)$ is obtained from that of its quotient by $Q(x)$ by multiplication by $Q(x)$. This finishes the proof of existence.

Uniqueness of the factorization is proved in the standard way just as the unqiueness of prime factorization for integers. □

The crucial result that underlies the method of partial fractions is this:

[t:crt]

Theorem 5.2.2. (Chinese Remainder Theorem) *Let $K(x)$ and $L(x)$ be relatively prime polynomials of degrees k and ℓ respectively. Given a polynomial $f(x)$ of degree less than $k + \ell$, there exist unique polynomials $S(x)$ and $T(x)$ of degress less than k and ℓ respectively, such that*

$$\frac{f(x)}{K(x)L(x)} = \frac{S(x)}{K(x)} + \frac{T(x)}{L(x)} \qquad \text{or, equivalently,} \qquad f(x) = S(x)L(x) + T(x)K(x) \tag{68}$$

Proof. Suppose that $S_1(x)$ and $T_1(x)$ also have the desired properties. Then, we have $(S(x) - S_1(x))L(x) = (T_1(x) - T(x))K(x)$. Since $L(x)$ and $K(x)$ are coprime, $K(x)$ divides $S(x) - S_1(x)$. But since $S(x) - S_1(x)$ has degree less than k, it follows that it is zero. So $S(x) = S_1(x)$. Similarly $T(x) = T_1(x)$ and the uniqueness is proved.

Since $K(x)$ and $L(x)$ are coprime polynomials, there exist $A(x)$ and $B(x)$ such that $A(x)K(x) + B(x)L(x) = 1$. Divide $f(x)B(x)$ by $K(x)$ and let $S(x)$ be the remainder. Divide $f(x)A(x)$ by $L(x)$ and let $T(x)$ be the remainder.

Clearly $S(x)$ and $T(x)$ have degrees less than k and ℓ respecively. Consider $f(x) - S(x)L(x) - T(x)K(x)$. This has degree less than $k + \ell$. We claim that it is divisible by

[6] In case $D(x) = 1$, the right hand side is an empty product, and empty products are by convention 1.

$K(x)$. Since the last term is divisible by $K(x)$, it is enough to show that $f(x) - S(x)L(x)$ is divisible by $K(x)$. In turn, it is enough to show that $f(x)B(x) - S(x)L(x)B(x)$ is divisible by $K(x)$, since $K(x)$ and $B(x)$ are relatively prime. Since $B(x)L(x) = 1 - A(x)K(x)$, we have $f(x)B(x) - S(x)L(x)B(x) = f(x)B(x) - S(x)(1 - A(x)K(x)) = (f(x)B(x) - S(x)) + A(x)K(x)$. The first parenthetical term on the right is divisible by $K(x)$ by the choice of $S(x)$, and the second is evidently divisible by $K(x)$. Our claim is thus proved.

In a similar fashion, we can prove that $f(x) - S(x)L(x) - T(x)K(x)$ is divisble by $L(x)$. Being divisible both by $L(x)$ and $K(x)$, it is divisible also by their product $K(x)L(x)$. But being of degree less than their product, it equals zero: that is $f(x) = S(x)L(x) + T(x)L(x)$. The theorem is thus proved. □

[ss:puthread]

5.3. **Expression as partial fractions of a proper fraction.** Let $N(x)/D(x)$ be a proper fraction with $D(x)$ non-zero and monic. Consider the factorization (67) of $D(x)$. Let $K(x)$ denote the highest power of one of the irreducible factors that divides $D(x)$ (say $(x-a)^{k_1}$ if $r \neq 0$, $Q_1^{\ell_1}$ otherwise) and $L(x)$ the product of the remaining factors, so that $D(x) = K(x)L(x)$. Apply Theorem 5.2.2 to get

$$\frac{N(x)}{D(x)} = \frac{S(x)}{K(x)} + \frac{T(x)}{L(x)} \tag{69}$$

Repeat the process with $L(x)$ and $T(x)$ in place of $D(x)$ and $N(x)$. Iterating this process as many times as required, we get

$$\frac{N(x)}{D(x)} = \frac{A_1(x)}{(x-a_1)^{k_1}} + \cdots + \frac{A_r(x)}{(x-a_r)^{k_r}} + \frac{B_1(x)}{Q_1^{\ell_1}} + \cdots + \frac{B_s(x)}{Q_s^{\ell_s}} \tag{70}$$

The polynomials $A_1(x), \ldots A_r(x)$, $B_1(x), \ldots, B_r(x)$ are uniquely determined of degrees less than $k_1, \ldots, k_r, 2\ell_1, \ldots, 2\ell_s$ respectively.

Now we apply the Taylor expansions to the polynomials A_i and B_j (see Exericses 2, 3):

$$A_i(x) = c_0^i + c_1^i(x-a_i) + c_2^i(x-a_i)^2 + \cdots + c_{k_1-1}^i(x-a_i)^{k_i-1} \tag{71}$$

$$B_j(x) = \mathfrak{c}_0^j + \mathfrak{c}_1^j Q_j + \mathfrak{c}_2^j Q_j^2 + \cdots + \mathfrak{c}_{\ell_1-1}^j Q_j^{\ell_1-1} \tag{72}$$

Dividing (71) by $(x-a_i)^{k_i}$ and (72) by $Q_j^{\ell_j}$, and substituting into (70), we get, finally, the expression as a sum of partial fractions for the proper fraction $N(x)/D(x)$:

$$\frac{N(x)}{D(x)} = \boxed{\sum_{1\leq i\leq r}\sum_{0\leq p\leq k_i-1} \frac{c_p^i}{(x-a_i)^{k_i-p}} + \sum_{1\leq j\leq s}\sum_{0\leq q\leq \ell_j-1} \frac{\mathfrak{c}_q^j}{Q_j^{\ell_j-q}}} \tag{73}$$

[ss:integrate]

5.4. **Anti-derivatives of partial fractions.** Thanks to (73), to integrate a proper fraction $N(x)/D(x)$ with $D(x)$ non-zero and monic, it is enough to know how to integrate $1/(x-a)^k$ and $(bx+c)/(x^2+1)^k$ (since any quadratic factor Q_j in (67) has no real roots, it may, after a linear change of variables and up to a non-zero factor, be written in the form (x^2+1)). The anti-derivative (always up to addition of a constant) of $1/(x-a)^k$ is $\log|x-a|$ in case $k=1$, and $-1/(k-1)(x-a)^{k-1}$ if $k \geq 2$. The anti-derivative of $x/(x^2+1)$ is $\log(x^2+1)/2$, of $x/(x^2+1)^k$ is $-1/2(k-1)(x^2+1)^{k-1}$ if $k \geq 2$. Finally, the anti-derivative of $1/(x^2+1)$ is

$\tan^{-1} x$ and that of $1/(x^2+1)^k$ for $k \geq 2$ may be found by the substitution $x = \tan y$ which reduces us to the integration of $\cos^{2(k-1)} y$ (with respect to y).

[s:exercises]

6. Exercises

Throughout, m stands for a positive integer. Solutions are available on the version of these notes on the home page of the author at `http://www.imsc.res.in/~knr/`

(1) Show that there is a unique polynomial of degree less than m that takes on specific values at m given values of the argument. With reference to the question raised in (11), the uniqueness part of this result is implicitly assumed in the method of plugging in values but not in the method of equating coefficients.

Solution: The existence of such a polynomial has been proved by Lagrange interpolation in §2.4. In the first item below we give a proof of uniqueness. In the second item below we give another proof which also proves existence at the same time.

(a) (A proof of uniqueness) Given two polynomials $f(x)$ and $g(x)$ both having the desired properties, consider their difference. It has degree at most m and vanishes at the m distinct values of the argument, which let us denote $a_1, \ldots, a_m$. This means that the difference polynomial is divisible by each of $x - a_1, \ldots, x - a_m$. These linear factors being pairwise coprime, the polynomial is divisible by their product, which evindently has degree m. Since the only polynomial of degree less than m that is divisible by a polynomial of degree m is the zero polynomial, it follows that $f(x) - g(x)$ is zero. $\square$

(b) (Another proof of uniqueness and existence) This uses the *van der Monde* matrix. A polynomial of degree less than m has the general form $f(x) = c_1 + c_2x + c_3x^2 + \cdots + c_m x^{m-1}$. To say that it has specified values $v_1, \ldots, v_m$ at m distinct values $a_1, \ldots, a_m$ of x means that:

$$f(a_i) = v_i = c_1 + c_2 a_i + c_3 {a_i}^2 + \cdots + c_m {a_i}^{m-1} \quad \text{for } 1 \leq i \leq m \tag{74}$$

These conditions may be written as a single matrix equation:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_m & a_m^2 & \ldots & a_m^{m-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \tag{75}$$

The $m \times m$ matrix in the above equation is called the *van der Monde* matrix. If we show that it is invertible, in other words, that it has non-zero determinant, then it follows that there is a unique set of coefficients $c_1, \ldots, c_m$ satisfying the above equation, and we are done.

We claim that the determinant of the van der Monde matrix is given by

$$\begin{array}{cccccccc} (a_2 - a_1) & (a_3 - a_1) & \cdots & \cdots & (a_{m-1} - a_1) & (a_m - a_1) & \cdot \\ & (a_3 - a_2) & \cdots & \cdots & (a_{m-1} - a_2) & (a_m - a_2) & \cdot \\ & & \cdots & \cdots & \cdots & \cdots & \cdot \\ & & & \cdots & \cdots & \cdots & \cdot \\ & & & & (a_{m-1} - a_{m-2}) & (a_m - a_{m-2}) & \cdot \\ & & & & & (a_m - a_{m-1}) & \end{array} \tag{76}$$

Indeed, each of the terms in the standard expansion has degree $1+2+\cdots+(m-1) = m(m-1)/2$, so the determinant is a polynomial of degree at most $m(m-1)/2$ (total degree in the a_i). On the other hand, since the determinant vanishes if any two of the a_i are equal, it is divisible by each of the linear factors $(a_i - a_j)$, $1 \leq j < i \leq m$. Thus it is a constant multiple of the product (76) and the constant of multiplication can be determined to be 1 by examining the coefficient of some term, e.g., $a_2 a_3^2 \cdots a_m^{m-1}$. $\square$

(2) Let $f(x)$ be a polynomial and a a constant. Then there exists a unique expression (meaning there is a unique sequence of coefficients $c_0, c_1, \ldots$) of the following form, in which only finitely many coefficients c_i are non-zero:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_i(x-a)^i + \cdots \tag{77}$$

This expression is called the *Taylor series expansion* of $f(x)$ around a. The largest i such that c_i does not vanish is the degree of $f(x)$. The coefficient c_i is the i^{th} derivative of $f(x)$ divided by $i!$ evaluated at a.

Solution: To prove uniqueness, suppose that there were two expressions for the same polynomial. Then the difference of the two expressions would on the one hand be zero as a polynomial; on the other, it would have degree equal to the largest integer such that the sequences of coefficients of the two expressions do not match at that point. We are thus lead to a contradiction. Alternatively, just observe that the coefficient c_i is forced to be the i^{th} derivative of $f(x)$ divided by $i!$ evaluated at a.

For existence, proceed by induction on the degree of $f(x)$. If $f(x)$ is of degree 0 then it is a constant and we just take c_0 to be this constant and the remaining c_i to be zero. Suppose now that $f(x)$ has positive degree. We take c_0 to be $f(a)$. Since $f(x) - c_0$ vanishes at a, it is divisible by $x - a$, and the quotient $g(x)$ having degree 1 less than f has an expression as above, multiplying which by $x - a$, we get the expression for $f(x)$: $f(x) = c_0 + (x-a) \cdot$ expression for $g(x)$. $\square$

(3) This generalizes the Taylor expansion of Exercise 2. The polynomial $Q(x)$ below of degree 2 could be replaced by one of higher degree and the conclusion would hold with appropriate modifications. Let $f(x)$ be a polynomial and $Q(x)$ a quadratic polynomial (with coefficient of x^2 non-zero). Then there exists a unique expression (meaning there is a unique sequence of linear forms $\mathfrak{c}_0, \mathfrak{c}_1, \ldots$) of the following form, in which only finitely many linear forms $\mathfrak{c}_i$ are non-zero:

$$f(x) = \mathfrak{c}_0 + \mathfrak{c}_1 Q + \mathfrak{c}_2 Q^2 + \cdots + \mathfrak{c}_i Q^i + \cdots \tag{78}$$

The largest i such that $\mathfrak{c}_i$ does not vanish is such that $\deg f(x)$ is either $2i$ or $2i+1$.

Solution: To prove uniqueness, suppose that there were two expressions for the same polynomial. Then the difference of the two expressions would on the one hand be zero as a polynomial; on the other, it would have degree equal to $2i$ or $2i+1$ where i is the largest integer such that the sequences of linear forms of the two expressions do not match at that point. We are thus lead to a contradiction.

For existence, proceed by induction on the degree of $f(x)$. We take $\mathfrak{c}_0$ to be the remainder when $f(x)$ is divided by $Q(x)$. If $f(x)$ is of degree at most 1, then we are done (by taking $\mathfrak{c}_i = 0$ for $i \geq 1$). If $\deg f(x) \geq 2$, let $g(x)$ be the quotient obtained when $f(x)$ is divided by $Q(x)$, and apply the induction hypothesis to $g(x)$. The expression for $f(x)$ is obtained from that of $g(x)$ as follows: $f(x) = \mathfrak{c}_0 + Q(x) \cdot \text{expression for } g(x)$. □

(4) Show that there is a unique polynomial of degree less than m all of whose derivatives of orders less than m take on specific values at a given value of the argument. (The derivative of zeroth order is the polynomial itself, so its value at the given value of the argument is in particular also specified.) The uniqueness part of this result is implicitly assumed in the method of repeatedly differentating and evaluating but not in the method of equating coefficients.

Solution: This follows from the statement of Exercise 2. Indeed, let a be the given value of the argument. In the right hand side of (77), let us put $A_i = 0$ for $i \geq m$ and equal to the specified value of the i^{th} derivative divided by $i!$. The resulting polynomial $f(x)$ has the desired properties.

To prove uniqueness, suppose that there were two polynomials both meeting the desired conditions. Their difference would be of degree less than m and all its derivatives of orders less than m would vanish at a. The Taylor series (77) of this polynomial would thus be zero, proving the equality of the original two polynomials. □

(5) In the Taylor expansion (77) of a polynomial $f(x)$ around any integer a, observe that if the coefficients of $f(x)$ are all integers, then so are all the coefficients A_i.

[x:litaylor]

(6) Let $d_1, \ldots, d_m$ be a sequence of positive integers and $a_1, \ldots, a_m$ a sequence of distinct real numbers. Show that there exists a unique polynomial of degree less than $d_1 + \cdots + d_m$ whose derivatives of all orders less than d_i have specified values at a_i, for every i, $1 \leq i \leq m$. The case when all the d_i are 1 is Lagrange interpolation (Exercise 1 and §2.4) and the case when $m = 1$ is the Taylor expansion (Exercise 2).

Solution: To prove uniqueness, suppose that f and g are two polynomials that meet the desired requirements, consider their difference. Since $f - g$ has derivatives of all orders less than d_i vanishing at a_i, it is divisible by $(x - a_i)^{d_i}$ (by Exercise 2). Since the $(x - a_i)^{d_i}$ as i varies are all pairwise coprime, their product also divides $f - g$. But then, since $f - g$ has degree less than $d_1 + \cdots + d_m$, it is zero.

To prove existence, we make an observation. Let $H(x)$ and $F(x)$ be polynomials, let a be a real number such that $H(a) \neq 0$, and let p be a positive integer. The derivatives of orders less than p of $F(x)$ and $G(x) = F(x)/H(x)$ are related thus:

$$\begin{gathered} F \;=\; GH \qquad F' \;=\; GH' \;+\; G'H \\ F^{(2)} \;=\; GH^{(2)} \;+\; 2G'H' \;+\; G^{(2)}H \end{gathered} \tag{79}$$

and more generally

$$F^{(k)} \;=\; \sum_{0 \leq j \leq k} \binom{k}{j} H^{(k-j)} G^{(j)} \qquad \text{for } 0 \leq k < p \tag{80}$$

Thus to specify the values at a of $(F(x)/H(x))^{(j)}$, for all $0 \leq j < p$, is equivalent to specify the values at a of $F(x)^{(j)}$ for all $0 \leq j < p$, for fixed $H(x)$ with $H(a) \neq 0$. Indeed, the equations above say the following: if we think of the two sets of values as $p \times 1$ column matrices, they are related by a lower triangular $p \times p$ matrix with diagonal entries all being $H(a)$.

Thanks to the observation above, it suffices to show that existence of a polynomial $f(x)$ of degree less than $d_1 + \cdots + d_m$ with the following property: for every i, $1 \le i \le m$, the derivatives at a_i of $f(x)/h_i(x)$ of all orders less than d_i have arbitrarily specified values, where $h_i(x) = \prod_{p \ne i}(x - a_p)^{d_p}$. We claim that such an $f(x)$ is given by

$$f(x) \;=\; \sum_{i=1}^{m} \sum_{j=1}^{d_i} A_i^j \, \frac{p(x)}{(x - a_i)^j} \tag{81}$$

where $p(x) = \prod_{i=1}^{m}(x - a_i)^{d_i}$ and A_i^j is $(d_i - j)!$ times the desired value at a_i of the derivative of order $d_i - j$ of $f(x)/h_i(x)$.

Indeed, each term on the right hand side has degree less than $d_1 + \cdots + d_m$, and therefore so does $f(x)$. Dividing (81) by $h_i(x)$:

$$\frac{f(x)}{h_i(x)} \;=\; \sum_{p \ne i} \sum_{q=1}^{d_p} A_p^q \, \frac{(x - a_i)^{d_i}}{(x - a_p)^q} \;+\; \sum_{q=1}^{d_i} A_i^j (x - a_i)^{d_i - j} \tag{82}$$

Each term in the first sum on the right hand side is a product of $(x - a)^{d_i}$ with a regular function not vanishing at a, so all its derivatives of orders less than d_i are zero when evaluated at a. Thus

$$\frac{1}{(d_i - j)!} \left(\frac{f(x)}{h_i(x)} \right)^{(d_i - j)} (a_i) \;=\; A_i^j \tag{83}$$

and we are done. $\square$

(7) In the following expression, $f(x)$ is a polynomial of degree less than $r_1 + \cdots + r_n$:

$$\frac{f(x)}{(x - a_1)^{r_1} \cdots (x - a_n)^{r_n}} \;=\; \sum_{1 \le i \le n} \sum_{1 \le j \le r_i} \frac{A_i^j}{(x - a_i)^j}$$

Show that A_i^j is given by the following formula:

$$\text{with } F_i = \frac{f(x)}{(x - a_1)^{r_1} \cdots (x - a_{i-1})^{r_{i-1}} (x - a_{i+1})^{r_{i+1}} \cdots (x - a_n)^{r_n}}$$

$\boxed{A_i^j \;=\; \dfrac{F_i^{(r_i - j)}(a_i)}{(r_i - j)!}}$ where $F_i^{(r_i - j)}$ denotes the derivative of order $r_i - j$ of F_i.

(8) Suppose that a student of yours came up with the following "solution" to the problem of finding $\int dx/(x^2 + 1)$ (whereas the standard answer is $\tan^{-1} x + C$):

Put

$$\frac{1}{x^2 + 1} = \frac{1}{(x + i)(x - i)} = \frac{A}{x + i} + \frac{B}{x - i}$$

Multiplying by $(x + i)(x - i)$, and then plugging in successively $x = -i$ and $x = i$, we get $A = i/2$, $B = -i/2$. Thus

$$\int \frac{1}{x^2 + 1}\, dx \;=\; \int \frac{i}{2(x + i)}\, dx \;-\; \int \frac{i}{2(x - i)}\, dx$$

$$= \frac{i}{2}(\log(x + i) - \log(x - i)) + \text{constant} = \frac{i}{2} \log \frac{x + i}{x - i} + \text{constant}$$

Is this a valid answer? How would you react?

(9) Suppose that you are accosted by a "troublesome" student with the following "solution" to the very first example considered in this article. How would you react?:

To compute

$$\int \frac{1}{x^2 - 9}\, dx$$

put $x = 3iy$, so that $dx = 3idy$ and $x^2 = -9y^2$. Substituting these values,

$$\begin{aligned} \int \frac{1}{x^2 - 9}\, dx &= \int \frac{3i}{-9y^2 - 9}\, dy = \frac{1}{3i} \int \frac{1}{y^2 + 1}\, dy \\ &= \frac{1}{3i} \tan^{-1} y + \text{constant} = \frac{1}{3i} \tan^{-1}(\frac{x}{3i}) + \text{constant} \end{aligned}$$

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