Theorem 1 (Engel) *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms. Then V has a basis with respect to which the elements of $\mathfrak{g}$ are strictly triangular matrices. In particular, $\mathfrak{g}$ is nilpotent.*

Proof : Excluding the trivial case $V = 0$, it is enough to show that there exists $0 \neq v \in V$ such that $Xv = 0$ for every X in $\mathfrak{g}$. We proceed by induction on the dimension of $\mathfrak{g}$. The proof being clear when this is 0, we assume that $\dim \mathfrak{g} \geq 1$. We will show presently that there exists an ideal $\mathfrak{h}$ in $\mathfrak{g}$ of codimension 1. Assuming this for the moment, it follows from the induction hypothesis that the subspace $W := \{w \in V \mid Hw = 0 \text{ for all } H \text{ in } \mathfrak{h}\}$ is non-zero. Since $\mathfrak{h}$ is an ideal, W is $\mathfrak{g}$-invariant. Letting Y be any element in $\mathfrak{g} \setminus \mathfrak{h}$ and choosing $0 \neq v$ in W such that $Yv = 0$, we are done.

That an ideal $\mathfrak{h}$ of codimension 1 exists follows immediately from this claim:

> Any proper subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is contained in a subalgebra of one bigger dimension in which it is an ideal.

To prove the claim, let $\mathfrak{h}$ be any proper subalgebra. Since $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ and each element of $\mathfrak{g}$ acts nilpotently on V, each element of $\mathfrak{g}$ acts nilpotently on itself. In particular, each element of $\mathfrak{h}$ acts nilpotently on $\mathfrak{g}/\mathfrak{h}$. By the induction hypothesis, there exists $0 \neq X \in \mathfrak{g}/\mathfrak{h}$ such that $[\mathfrak{h}, X] \subseteq \mathfrak{h}$. Now $\mathfrak{h} \oplus \mathbb{C}X$ is a subalgebra in which $\mathfrak{h}$ is an ideal. $\Box$

Complements

Theorem 2 (Group theoretic version of Engel) *A group acting by unipotent automorphisms on a vector space leaves some non-zero vector invariant. In particular a group acting faithfully by unipotent endomorphisms on a vector space is nilpotent.*

Proof : Let G be the group and V the vector space. We may assume that V is a simple G-module and further that G is a subgroup of $GL(V)$ consisting of unipotent elements. We will show that G is trivial.

Fix $g \in G$ and write $g = 1 + n$. We will show that $n = 0$ by showing that $\operatorname{Tr} nt = 0$ for all t in $\operatorname{End} V$. By the density theorem or Burnside's Theorem, G is a spanning set for $\operatorname{End} V$, and so it is enough that $\operatorname{Tr} ng' = 0$ for all g' in G. The function $\operatorname{Tr} g'$ being a constant ($= \dim V$) on G, we have

$$\operatorname{Tr} ng' = \operatorname{Tr} gg' - \operatorname{Tr} g' = 0. \qquad \Box$$