THE INSTITUTE OF MATHEMATICAL SCIENCES

LIE ALGEBRAS ELECTIVE COURSE
SEPTEMBER–NOVEMBER 2010

FINAL EXAM

07 DECEMBER 2010, 1500 TO 1800 HRS, MATSCIENCE ROOM 217

Please hand in your paper no later than at 1800. Answer in the space provided. Sheets for rough work are provided separately and should not be handed in.

(1) Let R be a root system (not necessarily reduced or irreducible), and let α, β be non-proportional roots. Suppose $\exists t \in \mathbb{R}$ such that $\gamma := \beta + t\alpha \in R$. Prove that (a) $2t \in \mathbb{Z}$ and (b) if α is indivisible, then $t \in \mathbb{Z}$. *Hint: Analyse the α-chains through β and γ.*

(2) Let R be a reduced, irreducible root system, and let $(\cdot|\cdot)$ be a $W := W(R)$-invariant inner product on $V := \operatorname{span} R$. Let C be a chamber, and let $B(C)$ be the corresponding set of simple roots. Let $X^+ := \mathbb{R}_{\geq 0}(B(C)) \subset V$. Define a partial order $\succeq$ on V by setting $\lambda \succeq \mu$ if $\lambda - \mu \in X^+$.
(a) If $\mu \in V$, prove that $\exists \gamma \in W \cdot \mu$ (the Weyl group orbit of μ) such that $(\gamma|\alpha) \geq 0$ for all $\alpha \in B(C)$.
(b) Let $\lambda \in C$. Suppose $\mu \in V$ is such that $\lambda \succeq w\mu$ for all $w \in W$. Prove that $(\lambda|\lambda) \geq (\mu|\mu)$, and equality holds iff $\mu \in W \cdot \lambda$.

(3) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

be its root space decomposition with respect to a chosen maximal toral subalgebra $\mathfrak{h}$; here $R \subset \mathfrak{h}^*$ is the set of roots.

(a) If $\alpha, \beta \in R$ are such that $\alpha + \beta \in R$, prove that $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$.

(b) Let $C \subset \mathfrak{h}^*$ be a chamber, and let $R^+ := R^+(C)$ be the corresponding set of positive roots, and $B := B(C)$, the set of simple roots. Define $\mathfrak{n}^+ := \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$. Prove that $\mathfrak{n}^+$ is a Lie subalgebra of $\mathfrak{g}$.

(c) Prove that $\mathfrak{n}^+$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing $\bigoplus_{\alpha \in B} \mathfrak{g}_\alpha$.

(4) Let R be a reduced root system, and let $(\cdot|\cdot)$ be a $W(R)$-invariant inner product on $V := \operatorname{span} R$. A subset B of R is said to be a *basis of* R if there exists a chamber C for which $B = B(C)$ (the set of simple roots in $R^+(C)$). (a) Let $S \subset R$ be a *linearly independent* set such that $R \subset \mathbb{R}_{\geq 0}(S) \cup \mathbb{R}_{\leq 0}(S)$. Prove that S is a basis of R. (b) If B is a basis of R, show that $B^\vee := \{\alpha^\vee : \alpha \in B\}$ is a basis of $R^\vee$. (c) Suppose $B = \{\alpha_i : i = 1 \cdots l\}$ is a basis of R. Let $\alpha = \sum_{i=1}^{l} c_i \, \alpha_i \in R$. Prove that $\dfrac{c_i(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$.

(5) Let R be a reduced root system, and let α, β be non-proportional roots; recall that $n(\alpha, \beta) := \frac{2(\alpha|\beta)}{(\beta|\beta)}$.
(a) Let $w := s_\alpha s_\beta \in W(R)$ and $m := n(\alpha, \beta)\, n(\beta, \alpha)$. Prove that the order of w equals $2, 3, 4$ or 6 according as $m = 0, 1, 2$ or 3. (b) Let R be the root system with Dynkin diagram A_l ($l \geq 1$). Let $S = \{\alpha_i : i = 1 \cdots l\}$ be a *basis* of R (see problem 4). Define $c := s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l} \in W(R)$. Compute the order of c. *Note: While c depends on the choice of ordering of the elements of S, it is a fact (which you can use without proof) that different choices give rise to elements which are conjugate to each other; hence the order $o(c)$ in $W(R)$ is well defined.*